POLARIZATIONS OF EQUIGENERATED STRONGLY STABLE IDEALS

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Abstract. In this paper, we extend the results of Almousa, Fløystad, and Lohne ([AFL22]) which completely characterize polarizations of powers of the maximal ideal $(x_1,\dots,x_n)^d \subset k[x_1,\dots,x_n]$ to the setting of *strongly stable* monomial ideals. In particular, we give a necessary and sufficient criterion for determining when any polarization of a given strongly stable ideal is a separated model, and we reproduce (in the strongly stable case) [AFL22]'s classification of polarizations in terms of isotone maps. We also discuss conjectures and some results relating these polarizations to commutative algebra, simplicial topology, and algebraic geometry in the contexts of their associated Alexander duals, Stanley-Reisner simplicial complexes, and Hilbert schemes, respectively.

Contents

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1. Introduction

Monomial ideals play a central role in combinatorial commutative algebra, a field pioneered by the work of Stanley that connects commutative algebra and algebraic geometry to combinatorics on simplicial complexes. One of the main tools in their study is *polarization*, which was introduced in Hartshorne's thesis [Har66] in order to prove connectedness of the Hilbert scheme. Polarization replaces a monomial ideal with a square-free monomial ideal sharing the same homological data, giving access to combinatorial tools such as Stanley-Reisner theory.

Historically, polarization referred to the *standard polarization* (shown in the construction in the preceding paragraph), which is a specific method to separate the variables. But it turns out that many ideals have other, nonisomorphic polarizations. Another polarization which has been studied in some depth, the box polarization, was first introduced by Nagel and Reiner in [NR09] and further studied in [Yan12] as a tool to help construct the "complex of boxes." This "complex of boxes" is useful in that it gives a minimal, linear, cellular free resolution for the class of strongly stable ideals that we will study in this paper.

The aim of this paper is to extend the results of Almousa, Fløystad, and Lohne ([AFL22]) regarding polarizations of powers of the maximal ideal $(x_1,\dots,x_n) \subset k[x_1,\dots,x_n]$ to the class of strongly stable (monomial) ideals, and to explore the applications of this work. The class of strongly stable ideals, which generalizes the aforementioned family of powers of the maximal ideal, is defined by the following property: I is strongly stable if for any monomial $\mathbf{m} \in I$ and any x_i dividing $\mathbf{m}$, the monomials of the form $\mathbf{m}\cdot\frac{x_j}{x_i}$ are in I for all $1 \le j < i$. For example, if a strongly stable ideal contains the monomial $x_1x_2x_3^2$, it must also contain the monomial $x_1x_2^2x_3$ obtained by swapping an x_3 out for an x_2. Strongly stable ideals arise naturally in algebraic combinatorics; for instance, generic initial ideals are always strongly stable in characteristic 0.

The main result of this paper is Theorem 6.1 and its converse Proposition 4.2, a generalization of the spanning tree criterion for polarizations presented in the power of the maximal ideal case in [AFL22]. This allows us to determine combinatorially when a set of isotone maps defines a polarization for any strongly stable ideal, by looking at its graph of linear syzygies.

The paper is organized as follows. We begin with relevant background definitions and constructions that will define our setting in Section 2. We then move to Section 3, characterizing separated models of strongly stable ideals.

In Section 4, we introduce our definitions of down-triangles, and prove one direction of our main theorem. Section 5 explores a bootstrapping technique in the three variable case, and discusses why the strategy fails in higher dimensions. In Section 6, we prove the other direction of the main theorem by generalizing a strategy from [AFL22].

We conclude our paper with our work in three applications: in Section 7 we discuss Alexander duality and associated primes of polarizations, in Section 8 we explore the shellability of polarizations of strongly stable ideals, and in Section 9 we discuss when a polarization of a strongly stable ideal yields a smooth point on its associated Hilbert scheme.

Conventions. Unless otherwise stated, throughout the paper we will only consider *equigenerated* strongly stable ideals; that is, strongly stable ideals whose minimal monomial generators are all of the same degree $d \ge 2$. When $d = 1$, all strongly stable ideals are in the form of the homogeneous maximal ideal of a polynomial subring; hence the statements tend to be trivial. Unless otherwise mentioned, we will always be adopting this setup, so "strongly stable" will mean "strongly stable and equigenerated of degree ≥ 2".

2. Background

In this section we will present relevant definitions and constructions that will be useful to us throughout this paper.

2.1. **Strongly Stable ideals.**

Remark 2.1. Recall from Proposition 1.1.6 in Herzog-Hibi [HH11] that a monomial ideal has a unique minimal monomial set of generators. In the future, we mean a minimal set of generators whenever we refer to generators of a monomial ideal.

Definition 2.2. An *elementary move* (also known as a *Borel move*) e_{ij} where $i < j$ is the operation sending a monomial $\mathbf{m}$ to the monomial $\mathbf{m} \cdot \frac{x_i}{x_j}$, i.e.

$$e_{ij}(x_1^{a_1} \ldots x_n^{a_n}) = x_1^{a_1} \ldots x_i^{a_i+1} \ldots x_j^{a_j-1} \ldots x_n^{a_n}.$$

Such a move is *admissible* if $a_j \geq 1$ (that is, $e_{ij}(x_1^{a_1} \ldots x_n^{a_n})$ really is in our polynomial ring). We say a monomial m' is *reachable* from m if m' can be obtained from m by a sequence of admissible elementary moves. We say m is *maximal* if it is not reachable from any other generator of I.

Example 2.3. The monomial $x_1^3 x_2$ is reachable from $x_1 x_3^3$ through the sequence of admissible elementary moves $e_{13}e_{13}e_{23}$.

Notice that a monomial with exponent vector $(a_1, \ldots, a_n)$ is reachable from $(b_1, \ldots, b_n)$ exactly when $\sum_{i=1}^{j} a_i \geq \sum_{i=1}^{j} b_i$ for all $1 \leq j \leq n$.

Definition 2.4. Let $S = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k. A monomial ideal $I \subset S$ is *strongly stable* if for any generator $\mathbf{m}$ of I, I contains every monomial that is reachable from $\mathbf{m}$.

2.2. **Polarizations and separations.**

In this subsection, we recall the definitions of polarization and separation from [AFL22] and recall their characterization of polarizations of powers of the graded maximal ideal.

Notation 2.5. If R is a set, let $k[x_R]$ be the polynomial ring in the variables x_r where $r \in R$. If $S \to R$ is a map of sets, it induces a k-algebra homomorphism $k[x_S] \to k[x_R]$ by mapping x_s to x_r if $s \mapsto r$.

Definition 2.6 (Separation, Separated Model)**.** Let $R' \xrightarrow{p} R$ be a surjection of finite sets such that $|R'| = |R| + 1$. Let r_1 and r_2 be the two distinct elements of R' which map to a single element r in R. Let I be a monomial ideal in the polynomial ring $k[x_R]$ and J a monomial ideal in $k[x_{R'}]$. Then J is a *simple separation* of I if the following holds:

i. The monomial ideal I is the image of J by the map $k[x_{R'}] \to k[x_R]$.

ii. Both the variables x_{r_1} and x_{r_2} occur in some minimal generators of J (usually in distinct generators).

iii. The variable difference $x_{r_1} - x_{r_2}$ is a non-zero divisor in the quotient ring $k[x_{R'}]/J$.

More generally, if $R' \xrightarrow{p} R$ is a surjection of finite sets and $I \subseteq k[x_R]$ and $J \subseteq k[x_{R'}]$ are monomial ideals such that J is obtained by a succession of simple separations of I, then J is a *separation* of I. J a *separated model* (of I) if there are no possible nontrivial separations of J.

Definition 2.7 (Preseparation)**.** Define a *preseparation* of a monomial ideal $I \subset k[x_R]$ (using the same notation as Definition 2.6) to satisfy the same conditions as that definition, except (iii): that is, the appropriate difference of variables is not guaranteed to be a regular sequence.

Intuitively, one can think of a preseparation as a "potential" separation. Often in proofs we will put forth preseparations and need to check that they satisfy the "nonzero divisor" condition (iii) in Definition 2.6.

Definition 2.8. An ideal J is a *polarization* of an ideal I if it is a square-free separation of I.

Construction 2.9 (Standard Polarization)**.** Let I be a monomial ideal in the polynomial ring $S = k[x_1,\dots,x_n]$ over a field k. Let d_i be the largest power of the variable x_i which divides a minimal generator of I. Let Let $\check{X}_i = \{x_{i_1},\dots,x_{i_{d_i}}\}$ be a set of variables for each $i \in [n]$, and let $\tilde{S} = k[\check{X}_1,\dots,\check{X}_n]$ be a polynomial ring in the union of all these variables.

Take each generator of I of the form $x_1^{a_1}x_2^{a_2}\dots x_n^{a_n}$ and make the following monomial $(x_{11}x_{12}\dots x_{1a_1})\cdot(x_{21}x_{22}\dots x_{2a_2})\cdot\dots\cdot(x_{n1}x_{n2}\dots x_{na_n})$ a minimal generator of the ideal $\tilde{I} \subset \tilde{S}$.

Call $\tilde{I}$ the *standard polarization* of I. To recover the quotient ring S/I from $\tilde{S}/\tilde{I}$, quotient successively by the regular sequence of variable differences $x_{i1}-x_{i2},\dots,x_{i1}-x_{in}$ for each i.

Another useful polarization, which we refer to as the box polarization, was introduced by Nagel and Reiner in [NR09].

Construction 2.10 (Box Polarization)**.** Let I be a monomial ideal in the polynomial ring $S = k[x_1,\dots,x_n]$ over a field k. Let d_i be the largest power of the variable x_i which divides a minimal generator of I. Let Let $\check{X}_i = \{x_{i_1},\dots,x_{i_{d_i}}\}$ be a set of variables for each $i \in [n]$, and let $\tilde{S} = k[\check{X}_1,\dots,\check{X}_n]$ be a polynomial ring in the union of all these variables.

Take each generator of I of the form $x_1^{a_1}x_2^{a_2}\dots x_n^{a_n}$ and make the following monomial $(x_{11}x_{12}\dots x_{1a_1})\cdot(x_{2a_1+1}x_{2a_1+2}\dots x_{2a_1+a_2})\cdot\dots\cdot(x_{na_1+\dots+a_{n-1}+1}x_{na_1+\dots+a_{n-1}+2}\dots x_{na_1+\dots+a_n})$ a minimal generator of the ideal $\tilde{I} \subset \tilde{S}$.

Call $\tilde{I}$ the *box polarization* of I.

Notice that the second indices restart for each change in the first index of the standard polarization, while the second indices keep increasing in the box polarization.

Example 2.11. Observe:

The STANDARD POLARIZATION:

$$x_1^2x_2x_3^3 \mapsto x_{11}x_{12}x_{21}x_{31}x_{32}x_{33}$$

The BOX POLARIZATION:

$$x_1^2x_2x_3^3 \mapsto x_{11}x_{12}x_{23}x_{34}x_{35}x_{36}$$

This paper's aim is to shed light on more general polarizations.

For convenience, we record here a useful lemma and its corollary on square-free monomial ideals (Lemma 5.8 in [AFL22]):

Lemma 2.12. *Let I be a monomial ideal in $k[x_1,\dots,x_n]$ such that each generator of I is square-free in the x_i-variable. Then if $j \neq i$ and $(x_i - x_j)\cdot f$ is in I, then for every monomial $\mathbf{m}$ in f we have that $x_i\mathbf{m}$ and $x_j\mathbf{m}$ are in I.*

Lemma 2.13. *Let $\mathbf{x}^{\mathbf{a}}$ and $\mathbf{x}^{\mathbf{b}}$ be minimal generators of a monomial ideal I and $m(\mathbf{a})$ and $m(\mathbf{b})$ the corresponding generators in a polarization of I. Fix an index i. If $a_i \leq b_i$ and $a_j \geq b_j$ for every $j \neq i$, then the i'th part $m_i(\mathbf{a})$ divides $m_i(\mathbf{b})$.*

The lemma above is important in motivating the construction of isotone maps on a partial order encoding divisibility of the generators of a polarization, which will be crucial to the statement and proof of many of our results. We will now transition into providing background constructions and results from the original paper [AFL22].

The goal of the original paper was to study polarizations of powers of the maximal ideal, and while we concern ourselves with polarizations of strongly stable ideals, we will be borrowing much of the terminology and techniques introduced in [AFL22].

2.3. **Constructions and results from the literature.** In this section we summarize the characterization of polarizations of powers of the graded maximal ideal from [AFL22]

Notation 2.14. Fix integers n and d, and let $S = k[x_1,\dots,x_n]$ be a polynomial over a field k. Let $\check{X}_i = \{x_{i1},\dots,x_{id}\}$ be a set of variables, and let $\tilde{S} = k[\check{X}_1,\dots,\check{X}_n]$ be a polynomial ring in the union of all these variables. Denote by $\mathfrak{m} = (x_1,\dots,x_n)$ the graded maximal ideal of S.

Denote by $\Delta^{\mathbb{Z}}(n,d) = \Delta(n,d) \cap \mathbb{Z}^n$ the set of lattice points of the dilated simplex $d\cdot\Delta^{n-1}$, i.e., the set of tuples $\mathbf{a} = (a_1,\dots,a_n)$ of non-negative integers with $\sum_i^n a_i = d$. Consider the polytopal CW-complex with the underlying space $d\cdot\Delta_{n-1}$, with CW-complex structure induced by intersection with the cubical CW-complex structure on $\mathbb{R}^n$ given by the integer lattice $\mathbb{Z}^n$. Denote by $\mathcal{T}(n,d)$ the one-skeleton of this cell complex.

Observation 2.15. The elements of $\Delta^{\mathbb{Z}}(n,d)$ are exactly the exponent vectors of the minimal generating set of the ideal $\mathfrak{m}^d$.

Notation 2.16. Let $e_i \in \mathbb{N}^n$ be the ith unit vector in $\mathbb{N}^n$. For a given $\mathbf{a}$, denote by $\mathrm{Supp}(\mathbf{a})$ the *support* of $\mathbf{a}$, that is, the set of all i such that $a_i > 0$. If B is a subset of $[n]$, denote by $\mathbb{1}_B$ the n-tuple $\sum_{i\in B} e_i$. For example, if $B = [n]$, then $\mathbb{1}_B = (1,\dots,1)$.

In the following definitions, we recall from [AFL22] some key subgraphs of $\mathcal{T}(n,d)$ which will be critical for characterizing polarizations combinatorially.

Definition 2.17 (Complete down-graph)**.** Given $\mathbf{c} \in \Delta^{\mathbb{Z}}(n,d+1)$ and $i,j \in \mathrm{Supp}(\mathbf{c})$, there is an edge between $\mathbf{c} - e_i$ and $\mathbf{c} - e_j$ in $\mathcal{T}(n,d)$ denoted $(\mathbf{c};i,j)$. Every edge in $\mathcal{T}(n,d)$ can be realized as an edge $(\mathbf{c};i,j)$ for unique $\mathbf{c}, i$, and j. An n-tuple $\mathbf{c} \in \Delta^{\mathbb{Z}}(n,d+1)$ induces a subgraph of $\mathcal{T}(n,d)$ called the *complete down-graph* $D(\mathbf{c})$ on the points $\mathbf{c} - e_i$ for $i \in \mathrm{Supp}(\mathbf{c})$. If $R \subseteq [n]$, denote by $D_R(\mathbf{c})$ the complete graph with edges $(\mathbf{c};r,s)$ for $r,s \in R$.

Definition 2.18 (Complete up-graph)**.** Any $\mathbf{a} \in \Delta^{\mathbb{Z}}(n,d-1)$ also determines a subgraph of $\mathcal{T}(n,d)$: the *complete up-graph* $U(\mathbf{a})$ consisting of points $\mathbf{a} + e_i$ for $i = 1,\dots,n$ with edges $(\mathbf{a} + e_i + e_j;i,j)$ for $i \neq j$.

Remark 2.19. The complete down-graph $D(\mathbf{c})$ induces a simplex of full dimension $d-1$ if and only if $c_i \geq 1$ for all i, i.e., $\mathbf{c}$ has full support. For each $\mathbf{a}$ in $\Delta^{\mathbb{Z}}(n, d-1)$, the induced simplex of the up-graph $U(\mathbf{a})$ always has full dimension $d-1$.

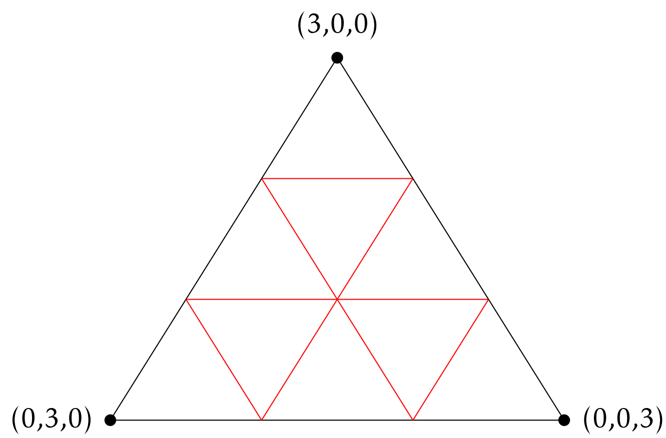


Figure 1. The graph $\mathcal{T}(3,3)$.

Example 2.20. The graph $\mathcal{T}(3,3)$ pictured in Figure 1 has three "complete down-triangles" with full support corresponding to the vectors $(2,1,1),(1,2,1)$, and $(1,1,2)$ in $\Delta^{\mathbb{Z}}(n, d+1)$. It also has six "complete up-triangles".

We now introduce a set of partial orders $\geq_i$ for each $i \in [n]$.

Definition 2.21 (The Partial Order $\geq_i$)**.** Adopt notation and hypotheses of Notation 2.14. Fix an index $1 \leq i \leq n$. Define $(\Delta^{\mathbb{Z}}(n,d), \geq_i)$ to be the poset with ground set $\Delta^{\mathbb{Z}}(n,d)$ and partial order $\geq_i$ such that $\mathbf{b} \geq_i \mathbf{a}$ if $b_i \geq a_i$ and $b_j \leq a_j$ for $j \neq i$.

Observation 2.22. The partial order $\geq_i$ as in Definition 2.21 is graded, where $\mathbf{a} \in \Delta^{\mathbb{Z}}(n,d)$ has rank a_i.

The maps in the following construction will play an important role in our efforts to combinatorially characterize polarizations throughout this paper.

Construction 2.23 (Isotone Maps)**.** Adopt notation and hypotheses of Notation 2.14. Let $\mathcal{B}_d$ be the Boolean poset on $[d]$ and $\{X_i\}_{1 \leq i \leq n}$ be a set of rank-preserving isotone maps

$$X_i : (\Delta^{\mathbb{Z}}(n,d), \leq_i) \to \mathcal{B}_d.$$

For any $\mathbf{a} \in \Delta^{\mathbb{Z}}(n,d)$, let $m_i(\mathbf{a}) = \prod_{j \in X_i(\mathbf{a})} x_{ij}$ and $m(\mathbf{a}) = \prod_{i=1}^{n} m_i(\mathbf{a})$. Let J be the ideal in $k[\check{X}_1, \ldots, \check{X}_n]$ generated by the $m(\mathbf{a})$.

Definition 2.24 (Linear Syzygy Edge)**.** Let $(\mathbf{c}; i, j)$ be an edge of $\mathcal{T}(n,d)$, where $\mathbf{c} \in \Delta^{\mathbb{Z}}(n, d+1)$. Then $(\mathbf{c}; i, j)$ is a *linear syzygy edge* (or *LS-edge*) if there is a monomial $\mathbf{m}$ of degree $d-1$ such that

$$m(\mathbf{c}-e_i) = x_{jr} \cdot \mathbf{m} \quad \text{and} \quad m(\mathbf{c}-e_j) = x_{is} \cdot \mathbf{m},$$

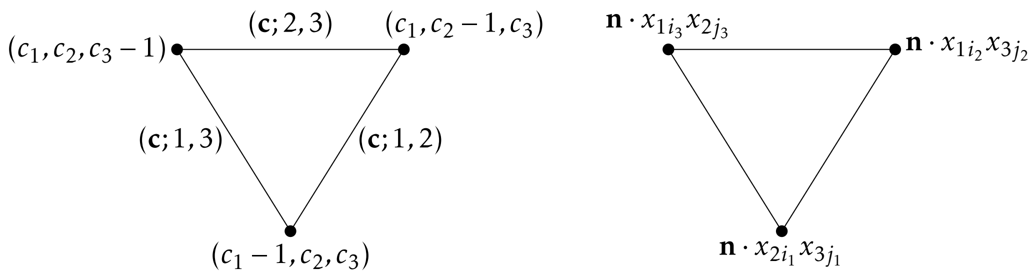


Figure 2. A down-triangle and its labeled monomials

for suitable variables $x_{jr} \in \check{X}_j$ and $x_{is} \in \check{X}_i$. This edge gives a linear syzygy between the monomials $m(\mathbf{c} - e_i)$ and $m(\mathbf{c} - e_j)$. Equivalently, in terms of the isotone maps,

$$X_p(\mathbf{c} - e_i) = X_p(\mathbf{c} - e_j)$$

for every $p \neq i, j$. Observe that both $m_i(\mathbf{c} - e_i)$ and $m_j(\mathbf{c} - e_j)$ are common factors of $m(\mathbf{c} - e_i)$ and $m(\mathbf{c} - e_j)$.

Sometimes, one may wish to consider whether two elements of $\Delta^{\mathbb{Z}}(n, d)$ would share a linear syzygy edge with respect to a *subset* of $[n]$.

Definition 2.25 (R-Linear Syzygy Edge)**.** Let $R \subseteq [n]$ and $\mathbf{c} \in \Delta^{\mathbb{Z}}(n, d+1)$ with R contained in the support of $\mathbf{c}$. Let $r, s \in R$. Define $(\mathbf{c}; r, s)$ to be an *R-linear syzygy edge* if

$$X_p(\mathbf{c} - e_r) = X_p(\mathbf{c} - e_s) \text{ for } p \in R \setminus \{r, s\}.$$

By the isotonicity of the X_p, for $p = r, s$,

$$X_r(\mathbf{c} - e_r) \subseteq X_r(\mathbf{c} - e_s), \quad X_s(\mathbf{c} - e_s) \subseteq X_s(\mathbf{c} - e_r).$$

Let $D_R(\mathbf{c})$ be the complete graph with edges $(\mathbf{c}; r, s)$ for $r, s \in R$.

The following lemma tells us that the monomials assigned to vertices of a down-triangle by a set of isotone maps must have a common factor which is easy to describe.

Lemma 2.26. *Let $\mathbf{c} \in \Delta^{\mathbb{Z}}(n, d)$ have support $C \subseteq \{1, 2, \ldots, n\}$. The monomials assigned to the vertices in the down-graph $D(\mathbf{c})$ by the maps X_i have a common factor of degree $\mathbf{c} - \mathbb{1}_C$. This common factor is $\prod_{i \in C} m_i(\mathbf{c} - e_i)$.*

Example 2.27. Let $m = 3$ and $\mathbf{c} = (c_1, c_2, c_3)$ be in $\Delta_3^+(n+1)$. On the left in Figure 2 is the down triangle $D(\mathbf{c})$. Let

$$\mathbf{n} = m_1(\mathbf{c} - e_1) \cdot m_2(\mathbf{c} - e_2) \cdot m_3(\mathbf{c} - e_3).$$

Then the monomials associated to the vertices of this down-triangle are shown to the right in Figure 2.

The following lemma turns out to be a useful tool for induction, and for applications in later sections.

Lemma 2.28. *Let $\mathbf{c} \in \Delta^{\mathbb{Z}}(n, d+1)$. If the set of linear syzygy edges in $LS(\mathbf{c})$ contains a spanning tree for $D(\mathbf{c})$, then for each $R \subseteq \mathrm{supp}(\mathbf{c})$, the set of R-linear syzygy edges contains a spanning tree for $D_R(\mathbf{c})$.*

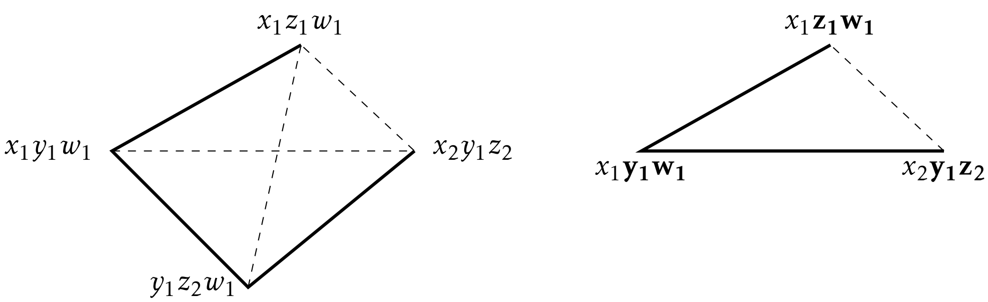


FIGURE 3. R-linear syzygy edges where $R = \{2, 3, 4\}$.

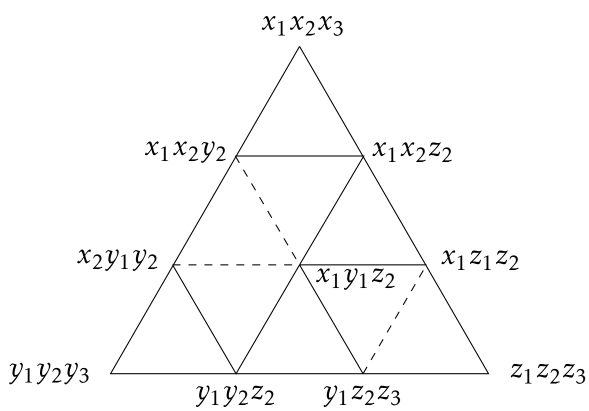


FIGURE 4. An example of a polarization of $(x, y, z)^3$.

Example 2.29. Consider the case of four variables and $\mathbf{c} = (1, 1, 1, 1)$. Write x, y, z, w for x_1, x_2, x_3, x_4, respectively. On the left of Figure 3 is the down-graph $D(\mathbf{c})$ with the three thick edges the linear syzygy edges.

Let $R = \{2, 3, 4\}$. On the right is the down-graph $D_R(\mathbf{c})$ where the two thick edges are the R-linear syzygy edges and the relevant variables marked in bold.

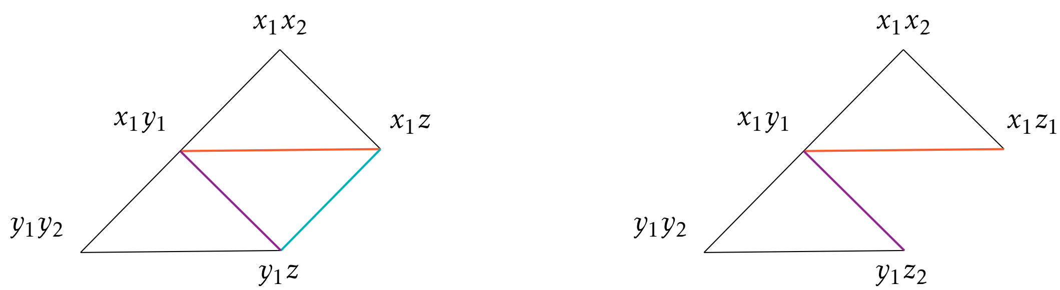
We conclude this section by recalling the main result of [AFL22] which offers a complete combinatorial characterization of all polarizations of $\mathfrak{m}^d$ in terms of their graphs of linear syzygies.

Theorem 2.30. *Adopt notation 2.14. A set of isotone maps $X_1, \ldots, X_n$ as in Construction 2.23 determines a polarization of the ideal $(x_1, \ldots, x_n)^d$ if and only if for every $\mathbf{c} \in \Delta^{\mathbb{Z}}(n, d+1)$, the linear syzygy edges $LS(\mathbf{c})$ contain a spanning tree for the down-graph $D(\mathbf{c})$.*

Example 2.31. Figure 4 depicts the graph of linear syzygies for a polarization of $(x, y, z)^3$. Notice that at most one edge is removed from each down-triangle, so it satisfies the spanning tree condition of Theorem 2.30.

3. Polarizations and Separated Models of Strongly Stable Ideals

In Section 2.2 of [AFL22], it is determined (Corollary 2.6) that any polarization of an Artinian monomial ideal is a separated model. Recall from Definition 2.6 that this means a polarization cannot be separated any further by "splitting" one of the variables that appear in it. However, this does not hold for strongly stable ideals, as the following example shows:



(A) The standard polarization of the ideal in Example 3.1.

(B) A separation of the standard polarization of the ideal in Example 3.1.

FIGURE 5. Two polarizations of a strongly stable ideal with linear syzygy edges marked, the second of which is a separation of the first.

Example 3.1. The standard polarization of the strongly stable ideal $(x^2, xy, xz, y^2, yz) \subset k[x, y, z]$ is *not* a separated model. The standard polarization is

$$I := (x_1x_2, x_1y_1, x_1z, y_1y_2, y_1z) \subset S' := k[x_1, x_2, y_1, y_2, z].$$

This has a further separation

$$J := (x_1x_2, x_1y_1, x_1z_1, y_1y_2, y_1z_2) \subset S'' := k[x_1, x_2, y_1, y_2, z_1, z_2].$$

Indeed, the natural surjection of polynomial rings sends J onto I, and if $f \in S''$ is such that $f(z_1 - z_2) \in J$, then Lemma 2.12 shows that for every monomial $\mathbf{m}$ of f, both $\mathbf{m}z_1$ and $\mathbf{m}z_2$ are in J. One sees from the generators of J that $\mathbf{m}z_1 \in J$ implies $x_1|\mathbf{m}$ or $\mathbf{m}$ is a multiple of a monomial generator of J, and that $\mathbf{m}z_2 \in J$ implies $y_1|\mathbf{m}$ or $\mathbf{m}$ is a multiple of a monomial generator of J. Since $x_1y_1 \in J$, it follows that $\mathbf{m} \in J$, so $f \in J$. Hence $z_1 - z_2$ is not a zero divisor in S''/J. See Figure 5 for the graphs of linear syzygies for the polarizations I and J.

It is therefore interesting to characterize the strongly stable ideals for which any polarization is a separated model. In particular, for strongly stable I, we will see that either any polarization of I is a separated model, or the standard polarization of I will not be a separated model.

For the setup, consider a function ρ, acting on the set of monomials as $\rho(x_1^{a_1} \dots x_n^{a_n}) = x_i^{a_i}$, where i is the largest index with a_i strictly positive. For instance, $\rho(x_1^5x_3x_4^2) = x_4^2$, regardless of what polynomial ring in which we consider the monomial $x_1^5x_3x_4^2$.

Lemma 3.2. *Suppose I is strongly stable with minimal generators $(\mathbf{a}_1, \dots, \mathbf{a}_k)$ each of degree d. Suppose there exists a maximal generator $\mathbf{a}_j$ such that there is a distinct generator $\mathbf{a}_k$ (not necessarily maximal) with $\rho(\mathbf{a}_j)|\mathbf{a}_k$. Then the standard polarization of I is not a separated model.*

Proof. Let $\mathbf{a}_j = x_1^{b_1} \dots x_i^{b_i}$ and $\mathbf{a}_k = x_1^{c_1} \dots x_n^{c_n}$, where $c_i \geq b_i \geq 1$. In the standard polarization $I' \subset S'$, the indeterminant x_{i,b_i} appears in at least two different monomials (preimages of $\mathbf{a}_j$ and $\mathbf{a}_k$). Consider the preseparation I'' inside $S'' := k[x_{1,1}, \dots, \widehat{x_{i,b_i}}, \dots, x_{n,e_n}, y_1, y_2]$, such that I'' has the same generators as I', but we replace every instance of x_{i,b_i} with y_1 *except* the x_{i,b_i} appearing in $m(\mathbf{a}_j)$, which we replace with y_2.

We claim that $y_1 - y_2$ is not a zero divisor in S''/I'', so this preseparation is actually a separation. Indeed, suppose $f \in S''$ with $f(y_1 - y_2) \in I''$. Since I'' is equigenerated of degree $d \geq 2$, $y_1 - y_2 \notin I''$. We want $f \in I''$, so it suffices to show that every monomial $\mathbf{m}$ of f is in I''. Since I'' is square-free, by Lemma 2.12, we must have $\mathbf{m}y_1, \mathbf{m}y_2 \in I''$. We see from the latter condition that either $\mathbf{m}$ is divisible by a generator of I'', or it is divisible by $x_{1,1} \dots x_{i,b_i-1}$, so consider the second case.

The indeterminant y_1 can only appear in a generator I'' as a result of pre-separating $m(\mathbf{a}_k)$, where $\mathbf{a}_k = x_1^{c_1} \dots x_n^{c_n}$, $\mathbf{a}_k \neq \mathbf{a}_j$, and $c_i \geq b_i$. Notice that it is not possible to have $c_{i'} \leq b_{i'}$ for all $1 \leq i' \leq i$, since $\sum b_l = d = \sum c_l$ and $b_i \leq c_i + \dots + c_n$, so this would contradict maximality of $\mathbf{a}_j$. Therefore pick i' with $c_{i'} > b_{i'}$.

The point is as follows. For $\mathbf{m}y_1$ to be in I'', $\mathbf{m}$ is either divisible by a generator of I'', or $\mathbf{m}y_1$ is divisible by some $x_{1,1} \dots \widehat{x_{i,b_i}} \dots x_{n,c_n} y_1$ induced by an $\mathbf{a}_k$ as above. So $\mathbf{m}$ is divisible by $x_{1,1} \dots \widehat{x_{i,b_i}} \dots x_{n,c_n}$, and in particular it is divisible by $x_{i',b_{i'}+1}$. Hence $x_{1,1} \dots x_{i,b_i-1} x_{i',b_{i'}+1} \mid \mathbf{m}$. But $x_1^{b_1} \dots x_{i'}^{b_{i'}+1} \dots x_i^{b_i-1}$ is in I because it is strongly stable, so $\mathbf{m} \in I''$. □

It turns out that the criterion given in Lemma 3.2 is in fact precisely the right condition needed to describe those strongly stable I whose polarizations are all separated models.

Lemma 3.3. *Suppose I is strongly stable with minimal generators $(\mathbf{a}_1, \dots, \mathbf{a}_k)$ each of degree d. Let $e_i \geq 1$ be the maximal power of x_i that appears in any $\mathbf{a}_j$. Suppose that for any maximal generator $\mathbf{a}_j$, the only generator divisible by $\rho(\mathbf{a}_j)$ is $\mathbf{a}_j$. Then any polarization of I is a separated model.*

Proof. Let $I' \subset S'$ be some polarization of $I \subset S$. Suppose that $I'' \subset S''$ is a further simple separation of I', so I'' is again a polarization of I.

First, if $\mathbf{a}_{j_i}$ is a maximal generator with $\rho(\mathbf{a}_{j_i}) = x_i^{b_i}$, then we must have $b_i = e_i$, and $\mathbf{a}_{j_i}$ is the unique generator divisible by $x_i^{e_i}$. This uniqueness also implies that $\mathbf{a}_{j_i} = x_1^{d-e_i} x_i^{e_i}$. Moreover, notice that we must have $d = e_1 \geq e_2 \geq \dots \geq e_n$ (if $e_i < e_j$ with $i > j$, then $x_1^{d-e_j} x_j^{e_j} \in I$ implies $x_1^{d-e_j} x_i^{e_j} \in I$, contradicting maximality of e_i).

We now claim that, if $i > 1$ and $\mathbf{a}_k = x_1^{c_1} \dots x_n^{c_n}$ is a generator divisible by x_i, then $c_1 \geq d - e_i$. If not, then $c_1 < d - e_i$, so $\mathbf{a}_k$ is not maximal as the only maximal generator divisible by x_i is $\mathbf{a}_{j_i} = x_1^{d-e_i} x_i^{e_i}$. But then $\mathbf{a}_k$ is not reachable from any of the maximal generators: surely it is not reachable from any of the $\mathbf{a}_{j_l}$ with $l \leq i$, and it is not reachable from any $\mathbf{a}_{j_l} = x_1^{d-e_l} x_i^{e_l}$ with $l > i$ as $c_1 < d - e_i \leq d - e_l$. This implies that $\mathbf{a}_k$ is maximal, a contradiction.

In particular, we see that if $\mathbf{a}_k$ is divisible by x_i (for any fixed $i > 1$), then $c_i < e_i$ and c_l is greater than or equal to the lth exponent of $\mathbf{a}_{j_i}$. It follows from Lemma 2.4 and Remark 2.5 in [AFL22] that if the polarization of $\mathbf{a}_{j_i}$ is $x_{1,1} \dots x_{1,d-e_i} x_{i,1} \dots x_{i,e_i}$ (after a possible re-indexing) inside I'', then every indeterminant of S'' mapping to x_i under a sequence of simple separations is one of the $\{x_{i,1}, \dots, x_{i,e_i}\}$. Then the proof of Corollary 2.6 in [AFL22] shows that the "split" indeterminant in the simple separation $S'' \twoheadrightarrow S'$ cannot be any of the $x_{i,l}$.

It remains to discuss x_1. But we know $\mathbf{a}_{j_1} = x_1^d$ is in I, so again Corollary 2.6 tells us that the "split" indeterminant in the simple separation $S'' \twoheadrightarrow S'$ cannot be any of the $x_{1,l}$. This contradicts the existence of the further simple separation I''. □

Note that an ideal satisfying the hypotheses of Lemma 3.3 need not be Artinian: consider $(x^2, xy, y^2, xz) \subset k[x, y, z]$.

4. Down-triangles in the Strongly Stable Case

Recall Theorem 2.30, which gives an explicit combinatorial description of all possible polarizations of a power of a maximal ideal $(x_1,\ldots,x_n)^d$:

Theorem 4.1. *Adopt notation 2.14. A set of isotone maps $X_1,\ldots,X_n$ as in Construction 2.23 determines a polarization of the ideal $(x_1,\ldots,x_n)^d$ if and only if for every $\mathbf{c} \in \Delta^{\mathbb{Z}}(n,d+1)$, the linear syzygy edges $LS(\mathbf{c})$ contain a spanning tree for the down-graph $D(\mathbf{c})$.*

We wish to extend this proposition to the case where the ideal in question is strongly stable. It turns out that the directly analogous criterion in the strongly stable case is the correct one:

Proposition 4.2. *Let $I \subseteq \Delta_n(d)$ correspond to the monomial generators of an ideal inside $k[x_1,\ldots,x_n]$ satisfying (*). Suppose that for every $\mathbf{c} \in \Delta_n(d+1)$, the set of linear syzygy edges $LS(\mathbf{c})$ arising from isotone maps $X_1,\ldots,X_n : I \to \mathcal{B}_d$ contains a spanning tree for the (partial) down-graph $D(\mathbf{c}) \cap I$. Then the X_i determine a polarization of I.*

We now recall the proof of Theorem 4.1 as outlined in Section 5 of [AFL22]. The first key ingredient is Lemma 2.28, which may be reformulated in our situation as follows:

Lemma 4.3. *Let $\mathbf{c} \in \Delta^{\mathbb{Z}}(n,d+1)$. If the set of linear syzygy edges in $LS(\mathbf{c})$ contains a spanning tree for $D(\mathbf{c}) \cap I$, then for each $R \subseteq \operatorname{supp}(\mathbf{c})$, the set of R-linear syzygy edges contains a spanning tree for $D_R(\mathbf{c}) \cap I$.*

Proof. Exactly as in Lemma 2.28. The only needed ingredient is that for any $\mathbf{a},\mathbf{b} \in D(\mathbf{c}) \cap I$, there is a path consisting of linear syzygy edges between $\mathbf{a}$ and $\mathbf{b}$. It does not matter that $D(\mathbf{c}) \cap I$ might be smaller than $D(\mathbf{c})$ (a point which we will return to many times). □

4.1. Linear Syzygy Paths.

It remains to discuss the main ingredient of the argument, which is a combinatorial lemma describing paths in $\Delta_n(d)$. We first have to make two definitions.

Definition 4.4. For points $\mathbf{a},\mathbf{b} \in \Delta_n(d)$, we write $\mathbf{a} \le \mathbf{b}$ if $a_i \le b_i$ for all i. This defines a partial order on $\Delta_n(d)$, and the join of $\mathbf{a}$ and $\mathbf{b}$ under this ordering is denoted

$$\mathbf{a} \vee \mathbf{b} := (\max(a_1,b_1),\ldots,\max(a_n,b_n)).$$

Definition 4.5. If $\mathbf{a} = (a_1,\ldots,a_n)$ and $\mathbf{b} = (b_1,\ldots,b_n)$ are two elements in $\Delta_n(d)$, with $A := \{i : a_i \ge b_i\}$ and $B := \{i : a_i < b_i\}$, then the *distance* between $\mathbf{a}$ and $\mathbf{b}$ is

$$d(\mathbf{a},\mathbf{b}) = \sum_{i\in A} a_i - b_i = \sum_{i \in B} b_i - a_i.$$

Intuitively, the distance between $\mathbf{a}$ and $\mathbf{b}$ is the minimum number of edges we need to use to construct a path from $\mathbf{a}$ to $\mathbf{b}$ within $\Delta_n(d)$.

With these definitions, we may state the main ingredient in the proof of Theorem 4.1. This is Proposition 5.3 in [AFL22]:

Proposition 4.6. *Let $\mathbf{a},\mathbf{b} \in \Delta_n(d)$. Suppose that for every $\mathbf{c} \in \Delta_n(d+1)$, the linear syzygy edges $LS(\mathbf{c})$ contains a spanning tree for the down-graph $D(\mathbf{c})$. Then there is a path*

$$\mathbf{a} = \mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_N = \mathbf{b}$$

such that

(1) Every $\mathbf{b}_i \leq \mathbf{a} \vee \mathbf{b}$.

(2) Every $m(\mathbf{b}_i)$ *divides the LCM of* $\mathbf{a}$ *and* $\mathbf{b}$.

(3) The edge from $\mathbf{b}_{i-1}$ *to* $\mathbf{b}_i$ *is an LS-edge.*

We call such a path an LS-path.

Once this result is known, Theorem 4.1 follows using Lemma 2.28. Observe that the proof of this last part does not need the fact that the ideal is a power of the maximal ideal. Therefore we want to modify this proposition to suit the strongly stable situation as follows:

Proposition 4.7. *Let* $I \subseteq \Delta_n(d)$ *correspond to the monomial generators of a strongly stable ideal inside* $k[x_1, \ldots, x_n]$*. Suppose that for every* $\mathbf{c} \in \Delta_n(d+1)$*, the set of linear syzygy edges* $LS(\mathbf{c})$ *arising from isotone maps* $X_1, \ldots, X_n : I \to \mathcal{B}_d$ *contains a spanning tree for the (partial) down-graph* $D(\mathbf{c}) \cap I$*. Then the conclusion to Proposition 4.6 holds inside* I*. In other words, for any* $\mathbf{a}, \mathbf{b} \in I$*, we may find a* restricted LS-path *between* $\mathbf{a}$ *and* $\mathbf{b}$*, which is an LS-path using only points in* I*.*

Let us follow the same method of proof as in [AFL22]. The proof of Proposition 4.6 has three parts, labeled A, B, and C. We aim to reproduce the argument of each part in our situation. In the following, we always assume $\mathbf{a}, \mathbf{b} \in I$.

Part A:

Lemma 4.8. *If* $d(\mathbf{a}, \mathbf{b}) = 1$*, then there is a restricted LS-path from* $\mathbf{a}$ *to* $\mathbf{b}$.

Proof. The proof follows the same outline as in [AFL22] (Lemma 5.4). □

We may now assume that $\mathbf{a}, \mathbf{b}$ have distance at least 2. Define

$$B := \{i : b_i > a_i\}, \quad A_> := \{i : a_i > b_i\}, \quad A_= := \{i : a_i = b_i\}.$$

Also let $A := A_> \cup A_=$. Now, write $P(\mathbf{b})$ for the set of all $\mathbf{b}' \in I$ such that:

(1) For $i \in B$, $\mathbf{b}$ and $\mathbf{b}'$ have equal ith coordinate.

(2) For $i \in A$, $b'_i \leq a_i$.

(3) There is a restricted LS-path from $\mathbf{b}'$ to $\mathbf{b}$ where the vertices $\mathbf{u}$ on the path satisfy $\mathbf{u} \leq \mathbf{a} \vee \mathbf{b}$ and $m(\mathbf{u}) \mid \operatorname{lcm}(m(\mathbf{a}), m(\mathbf{b}))$.

In particular, $\mathbf{b} \in P(\mathbf{b})$ (condition (3) becomes vacuous), so $P(\mathbf{b})$ is nonempty. Next, let $A_1 \subseteq A$ be the subset of all indices $i \in A$ for which there is some $\mathbf{b}' \in P(\mathbf{b})$ with $b'_i < a_i$, and let $A_0 := A - A_1$. Since $\mathbf{b} \in P(\mathbf{b})$, we get $A_> \subseteq A_1$, and in particular $A_>$ (hence A_1) is nonempty. We also conclude that $A_0 \subseteq A_=$, and moreover that $d(\mathbf{a}, \mathbf{b}) = d(\mathbf{a}, \mathbf{b}')$ for all $\mathbf{b}' \in P(\mathbf{b})$, via the second sum in Definition 4.5.

We now split into two cases to make a particular choice of $\beta \in B$. If $\max A_> > \min B$, then pick $\beta \in B$ such that there is $\alpha \in A_>$ with $\beta < \alpha$ (recall $A_> \subseteq A_1$). Since I is strongly stable, $\mathbf{a} \in I$ implies $\mathbf{a} + e_\beta - e_\alpha \in I$. Otherwise, we have $\max A_> < \min B$. Then for any $\beta \in B$ and $\alpha \in A_>$, it follows that $\mathbf{a} + e_\beta - e_\alpha$ is reachable from $\mathbf{b}$, hence is in I. Therefore with $R := A_1 \cup \{\beta\}$ for our choice of β, we know that $D_R(\mathbf{a} + e_\beta) \cap I$ contains points other than $\mathbf{a}$, and so by Lemma 4.3, there is an R-linear syzygy edge between $\mathbf{a}$ and $\mathbf{a} + e_\beta - e_\alpha$ for some $\alpha \in A_1$. The rest of the arguments for Part A in [AFL22] can be repeated, with "LS-path" replaced with "restricted LS-path".

Part B: The argument for Part B in [AFL22] can be repeated, with "LS-path" replaced with "restricted LS-path".

Part C: The argument for Part C in [AFL22] can be repeated, with "LS-path" replaced with "restricted LS-path".

This proves Proposition 4.7, and hence Proposition 4.2.

5. Extending Isotone Maps and Polarizations

It is also reasonable to ask the following question: given a subset $I \subseteq \Delta_n(d)$ corresponding to the monomial generators of a strongly stable ideal and isotone maps $X_1, \ldots, X_n : I \to \mathcal{B}_d$, when can we extend the X_i to all of $\Delta_n(d)$? Indeed, this is possible when $n = 3$.

Proposition 5.1. *Let $I \subseteq \Delta_3(d)$ correspond to the monomial generators of an ideal inside $k[x_1, x_2, x_3]$ satisfying (*). Suppose that for every $\mathbf{c} \in \Delta_3(n+1)$, the set of linear syzygy edges $LS(\mathbf{c})$ arising from isotone maps $X_1, X_2, X_3 : I \to \mathcal{B}_d$ contains a spanning tree for the (partial) down-graph $D(\mathbf{c}) \cap I$. Then the conclusion to Proposition 6.1 holds inside I. In other words, for any $\mathbf{a}, \mathbf{b} \in I$, we may find a* restricted LS-path *between $\mathbf{a}$ and $\mathbf{b}$, which is an LS-path using only points in I.*

Lemma 5.2. *Let $I \subseteq \Delta_3(d)$ correspond to the monomial generators of a strongly stable ideal inside $k[x_1, x_2, x_3]$. Suppose that for every $\mathbf{c} \in \Delta_3(d+1)$, the set of linear syzygy edges $LS(\mathbf{c})$ arising from isotone maps $X_1, X_2, X_3 : I \to \mathcal{B}_d$ contains a spanning tree for the (partial) down-graph $D(\mathbf{c}) \cap I$. These isotone maps X_1, X_2, X_3 from Proposition 6.4 can be extended to all of $\Delta_3(d)$ in a way such that the down-graph condition is preserved in the graph of linear syzygy edges for $\Delta_3(d)$.*

Proof. Induction on the size s of $\overline{I} := \Delta_3(d) - I$. Let $\mathbf{d} := (d_1, d_2, d_3)$ be in I' such that d_1 is minimal for $\{x_1 : (x_1, x_2, x_3) \in I'\}$, and d_3 is minimal for $\{x_3 : (d_1, x_2, x_3) \in I'\}$. There are three cases.

If $d_3 = 0$, then $(d_1+1, d_2-1, 0) \in I$, and we can extend X_1, X_2, X_3 to $\mathbf{d}$ as $X_1(\mathbf{d}) := X_1(d_1+1, d_2-1, 0) - \{i_1\}$ (for any i_1 in the d_1+1-element set $X_1(d_1+1, d_2-1, 0)$), $X_2(\mathbf{d}) := X_2(d_1+1, d_2-1, 0) \cup \{i_2\}$ (for any i_2 not in the d_2-1-element set $X_2(d_1+1, d_2-1, 0)$), and $X_3 := \emptyset$. The only new nonempty partial down-graph $D(\mathbf{c}) \cap (I \cup \mathbf{d})$ that does not appear in the set of $D(\mathbf{c}) \cap I$ is $\{(d_1, d_2, 0); (d_1+1, d_2-1, 0)\}$ (given by $\mathbf{c} = (d_1+1, d_2, 0)$), which is connected by construction.

If $d_2, d_3 \neq 0$, then (d_1, d_2+1, d_3-1), (d_1+1, d_2, d_3-1) and (d_1+1, d_2-1, d_3) are all in I. The new down-graphs are $\{\mathbf{d}, (d_1, d_2+1, d_3-1)\}$ and $\{\mathbf{d}, (d_1+1, d_2, d_3-1), (d_1+1, d_2-1, d_3)\}$, and by assumption, there is a linear syzygy between (d_1+1, d_2, d_3-1) and (d_1+1, d_2-1, d_3) since they are the members of $D(\mathbf{d}+e_1) \cap I$. In particular, X_1 is identical on those two points, so we define $X_1(\mathbf{d}) := X_1(d_1, d_2+1, d_3-1)$ (by isotonicity, $X_1(d_1, d_2+1, d_3-1)$ is a d_1-element subset of the d_1+1-element set $X_1(d_1+1, d_2, d_3-1) = X_1(d_1+1, d_2-1, d_3)$). Next, define $X_2(\mathbf{d}) := X_2(d_1+1, d_2, d_3-1)$, which, again by isotonicity, is a d_2-element subset of the d_2+1-element set $X_2(d_1, d_2+1, d_3-1)$ and a superset of $X_2(d_1+1, d_2-1, d_3)$. Finally, because the down-graph defined by (d_1+1, d_2+1, d_3-1) includes (d_1, d_2+1, d_3-1) and (d_1+1, d_2, d_3-1) and is contained in I, it follows that $X_3(d_1, d_2+1, d_3-1)$ and $X_3(d_1+1, d_2, d_3-1)$ differ by at most one element. So define $X_3(\mathbf{d})$ to be $X_3(d_1+1, d_2-1, d_3)$ if they are equal, and $X_3(d_1, d_2+1, d_3-1) \cup X_3(d_1+1, d_2, d_3-1)$ if they differ (one can again see that isotonicity is preserved). Notice that there is a linear syzygy between $\mathbf{d}$ and (d_1, d_2+1, d_3-1): the X_1 parts are equal by definition, the X_2 parts differ by 1 element by construction, and the X_3 parts differ by 1 element in either case (in particular, if we're in the $X_3(\mathbf{d}) = X_3(d_1+1, d_2-1, d_3)$ case, then $X_3(d_1+1, d_2, d_3-1)$ differs from $X_3(\mathbf{d})$ by 1 element due to isotonicity, and $X_3(d_1, d_2+1, d_3-1) = X_3(d_1+1, d_2, d_3-1)$). Finally, we show that the down-graph $\{\mathbf{d}, (d_1+1, d_2, d_3-1), (d_1+1, d_2-1, d_3)\}$ is connected; it suffices to check that there is a linear syzygy between $\mathbf{d}$ and (d_1+1, d_2, d_3-1). By definition, their X_2

parts are the same, and their X_1-parts differ by 1 element since $X_1(\mathbf{d}) = X_1(d_1, d_2+1, d_3-1)$. Their X_3-parts also differ by 1 element in either case, as before.

If $d_2 = 0$ but $d_2 \neq 0$, then the same extension procedure as in the above paragraph works; details omitted. In any case, we get an extension of our isotone maps to $\mathbf{d}$ such that the down-graph property is preserved in the extension. □

The above proof can be easily modified to remove the appearances of the down-graph condition in the hypothesis and conclusion of the lemma.

However, this extension property fails in higher dimensions. For instance:

Example 5.3. Consider the following subset I of $\Delta_4(3)$, which corresponds to monomial generators of a strongly stable ideal inside $k[x_1, x_2, x_3, x_4]$:

$$\{(1,0,1,1);(1,1,0,1);(2,0,0,1);(1,0,2,0);(1,1,1,0);(1,2,0,0);(2,0,1,0);(2,1,0,0);(3,0,0,0)\}.$$

Note that this corresponds to the strongly stable closure of the ideal generated by $x_1x_3x_4$. Now, define isotone maps $X_1, X_2, X_3, X_4 : I \to \mathcal{B}_3$ which map the above points to the following list of monomials:

$$x_{1,1}x_{3,1}x_{4,1}, \quad x_{1,1}x_{2,1}x_{4,2}, \quad x_{1,1}x_{1,2}x_{4,3}, \quad x_{1,1}x_{3,1}x_{3,2},$$
$$x_{1,1}x_{2,1}x_{3,1}, \quad x_{1,1}x_{2,1}x_{2,2}, \quad x_{1,1}x_{1,2}x_{3,1}, \quad x_{1,1}x_{1,2}x_{2,1}, \quad x_{1,1}x_{1,2}x_{1,3}.$$

One can easily check that the X_i's are isotone, and moreover the partial down-graph condition is satisfied (one can even check that this is a polarization, obtained from the standard polarization in the spirit of Lemma 4.10). The only cases we really need to check are the down-graphs corresponding to $(1,1,1,1)$, $(2,0,1,1)$, and $(2,1,0,1)$. For instance, we have $D(1,1,1,1) \cap I = \{(1,0,1,1);(1,1,0,1);(1,1,1,0)\}$, and there are linear syzygies between $(1,1,1,0)$ and the other two vertices, so the down-graph is connected (the other cases are similar). But there is no way to extend X_4 to $(1,0,0,2)$ in an isotone manner: its x_4-part must be divisible by $m_4(1,0,1,1) = x_{4,1}$, $m_4(1,1,0,1) = x_{4,2}$, and $m_4(2,0,0,1) = x_{4,3}$, which is clearly impossible.

5.1. An Alternate Proof of Proposition 4.7 in the Three Variable Case.

It turns out that we may use Lemma 5.2 to provide an alternate proof of Proposition 4.7 in the 3-variable case. This argument is interesting because it bootstraps off of the statement of Proposition 4.6, without ever making contact with its proof.

Lemma 5.4. *Let $I \subseteq \Delta_3(d)$ correspond to the monomial generators of an ideal inside $k[x_1, x_2, x_3]$ satisfying (*). Define the* boundary *B of I to consist of all $(d_1, d_2, d_3) \in I$ such that $d_1 = 0$ and $(d_1, d_2-1, d_3+1) \notin I$, or $(d_1-1, d_2, d_3+1) \notin I$. In other words, B is exactly the set of points in I that are distance 1 away from a point in I', due to the strongly stable condition.*

Let (a_1, a_2, a_3) and (b_1, b_2, b_3) be in B. If $a_3 < b_3$, then $a_1 \leq b_1$ (and consequently $a_2 > b_2$).

Proof. If $a_1 = 0$, then we are immediately done, so assume $a_1 \geq 1$. Now, if $a_1 > b_1$, then $a_1 - 1 \geq b_1$. Also, $a_3 + 1 \leq b_3$. Since I is strongly stable, $(b_1, b_2, b_3) \in I$ implies $(b_1, n - b_1 - (a_3+1), a_3+1) \in I$, and since $a_1 - 1 \geq b_1$, this implies $(a_1 - 1, a_2, a_3 + 1) \in I$, contrary to assumption. □

Lemma 5.5. *Let* $\mathbf{c}$ *be on an LS-path from* $\mathbf{a}$ *to* $\mathbf{b}$ *(here we do not need any assumptions about I). Then* $d(\mathbf{a}, \mathbf{c}) \leq d(\mathbf{a}, \mathbf{b})$.

Proof. Let $A_<$ be the set of i with $a_i < b_i$ and $A_\geq$ be the set of i with $a_i \geq b_i$. Also, let $C_>$ be the set of i with $c_i > a_i$. Then $C_>$ cannot intersect $A \geq$, for we would have $c_i > \max(a_i, b_i)$ for any i in the intersection. Hence $C_> \subseteq A_<$, and for $i \in C_>$, we must have $b_i \geq c_i$, so

$$d(\mathbf{a},\mathbf{c}) = \sum_{i \in C_>} c_i - a_i \leq \sum_{i \in C_>} b_i - a_i \leq \sum_{i \in A_<} b_i - a_i = d(\mathbf{a},\mathbf{b}).$$

□

Lemma 5.6. *Let $I \subseteq \Delta_3(d)$ correspond to the monomial generators of an ideal inside $k[x_1,x_2,x_3]$ satisfying (*), and let $\mathbf{a},\mathbf{b}$ be on the boundary B. If $\mathbf{c} \in B$ is on an LS-path from $\mathbf{a}$ to $\mathbf{b}$ and is not an endpoint, then $d(\mathbf{a},\mathbf{c}), d(\mathbf{b},\mathbf{c}) < d(\mathbf{a},\mathbf{b})$.*

Proof. Note that $c_1 \neq 0$, since there can only be one point in B with 0 in the first component.

First, consider the case $a_3 = b_3$, so without loss of generality, $a_1 < b_1$. If $c_3 = a_3 = b_3$, then we must have $a_1 < c_1 < b_1$ and $a_2 > c_2 > b_2$ (recall $a_1 + a_2 + a_3 = b_1 + b_2 + b_3 = c_1 + c_2 + c_3 = n$), so $d(\mathbf{a},\mathbf{c}) = c_1 - a_1 < b_1 - a_1 = d(\mathbf{a},\mathbf{b})$ and $d(\mathbf{b},\mathbf{c}) = c_2 - b_2 > a_2 - b_2 = d(\mathbf{a},\mathbf{b})$. If $c_3 < a_3$, then $c_3 + 1 \leq a_3$, and we must have $a_1 < c_1$ (if they were equal, then $c_2 > a_2$, which is a contradiction as $a_2 > b_2$). So $(a_1, n - a_1 - (c_3+1), c_3+1) \in I$ and $(c_1 - 1, c_2, c_3 + 1) \in I$ by the strongly stable condition, contradicting $\mathbf{c} \in B$.

Otherwise, consider the case $a_3 < b_3$. From Lemma 5.4 we know that $a_1 \leq b_1$ and $a_2 > b_2$, so $d(\mathbf{a},\mathbf{b}) = a_2 - b_2$. We claim that $a_3 \leq c_3 \leq b_3$. If not, then $c_3 < a_3$, and we must have $a_1 < c_1$ (if they were equal, then $c_2 > a_2$, which is a contradiction as $a_2 \geq b_2$), giving a contradiction to $\mathbf{c} \in B$ as above. Now, if $a_3 = c_3$, then we again must have $a_2 > c_2$ and $a_1 < c_1 \leq b_1$, and Lemma 5.4 (applied to $c_3 = a_3 < b_3$) tells us that $c_2 > b_2$. Hence $d(\mathbf{a},\mathbf{c}) = a_2 - c_2 < a_2 - b_2 = d(\mathbf{a},\mathbf{b})$ and $d(\mathbf{b},\mathbf{c}) = c_2 - b_2 < a_2 - b_2 = d(\mathbf{a},\mathbf{b})$. Otherwise, $a_3 < c_3$, so Lemma 5.4 (applied to $a_3 < c_3$) tells us that $a_1 \leq c_1$ and $a_2 > c_2$. The first inequality implies $c_1 \leq b_1$, so $d(\mathbf{b},\mathbf{c}) = c_2 - b_2 < a_2 - b_2 = d(\mathbf{a},\mathbf{b})$. We also claim that $c_2 > b_2$. If not (so $c_2 \leq b_2$), then $c_3 \leq b_3$ along with Lemma 5.4 forces $b_3 = c_3$, so $c_1 \geq b_1$, implying $\mathbf{b} = \mathbf{c}$, contrary to assumption. Hence $a_2 > c_2 > b_2$ and $d(\mathbf{a},\mathbf{c}) = a_2 - c_2 < a_2 - b_2 = d(\mathbf{a},\mathbf{b})$. □

Proof of Proposition 4.7, $n = 3$. We are now in a position to prove Proposition 4.7 when $n = 3$. First, by Lemma 5.2, we may extend X_1, X_2, X_3 to all of $\Delta_3(d)$ in a way preserving the down-graph condition. We claim that it suffices to verify that there is a restricted LS-path between any $\mathbf{a}$ and $\mathbf{b}$ on the boundary B. Indeed, if this is true, then for any other $\mathbf{a}', \mathbf{b}' \in I$, we know from Proposition 4.6 that there is an LS-path in $\Delta_3(d)$ between $\mathbf{a}'$ and $\mathbf{b}'$, possibly using points outside of I. But if this path leaves I, then it must contain boundary points $\mathbf{b}_i$ and $\mathbf{b}_j$ such that for all $k \leq i$ and $k \geq j$, $\mathbf{b}_k \in I$. By assumption, there is a restricted LS-path between $\mathbf{b}_i$ and $\mathbf{b}_j$, so we can "patch in" this restricted path to our original LS-path, which now looks like

$$\mathbf{a}' = \mathbf{b}_0, \ldots, \mathbf{b}_i, \text{restricted LS-path}, \mathbf{b}_j, \ldots, \mathbf{b}_N = \mathbf{b}'.$$

This path lies entirely in I by construction, and consists of linear syzygy edges. Moreover, for any $\mathbf{c}$ in the restricted LS-path, we have $\mathbf{c} \leq \mathbf{b}_i \vee \mathbf{b}_j \leq \mathbf{a}' \vee \mathbf{b}'$, and $m(\mathbf{c}) \mid \operatorname{lcm}(m(\mathbf{a}'), m(\mathbf{b}'))$. Note that the latter claim is true because $m(\mathbf{c}) \mid \operatorname{lcm}(m(\mathbf{b}_i), m(\mathbf{b}_j))$, and $m(\mathbf{b}_i), m(\mathbf{b}_j) \mid \operatorname{lcm}(m(\mathbf{a}'), m(\mathbf{b}'))$ since they were already on the original LS-path from $\mathbf{a}'$ to $\mathbf{b}'$. So this path is indeed a restricted LS-path from $\mathbf{a}'$ to $\mathbf{b}'$.

Now, assume for the sake of contradiction that there is a pair of boundary points with no restricted LS-path between them. We may choose such a pair $\mathbf{a}, \mathbf{b} \in B$ with minimal distance among all such pairs (if one is worried about edge cases, we may assume that this distance is at least 2 by Lemma 4.8). Consider an LS-path

$$\mathbf{a} = \mathbf{b}_0, \dots, \mathbf{b}_N = \mathbf{b}$$

in $\Delta_3(d)$ between $\mathbf{a}$ and $\mathbf{b}$. We claim that for all $1 \le i \le N-1$, $\mathbf{b}_i \notin I$. If not, then there is some $\mathbf{b}_i \in B$ not equal to one of the endpoints, and by Lemma 5.6, $d(\mathbf{b}_i, \mathbf{a}), d(\mathbf{b}_i, \mathbf{b}) < d(\mathbf{a}, \mathbf{b})$. By minimality there are restricted LS-paths from $\mathbf{a}$ to $\mathbf{b}_i$ and $\mathbf{b}_i$ to $\mathbf{b}$, and patching them together gives a restricted LS-path from $\mathbf{a}$ to $\mathbf{b}$, contrary to our assumption.

We again split into cases. First, consider the case $b_3 = a_3$ (so we can assume b_3 is nonzero, otherwise we reduce to the case where $\mathbf{a}$ and $\mathbf{b}$ lie on the xy-edge, which is trivial). Assume without loss of generality that $b_1 > a_1$ (so b_1 is nonzero). By the strongly stable condition, $(b_1 - 1, b_2 + 1, b_3) \in I$ (since $a_1 \le b_1 - 1$ and $(a_1, a_2, a_3) \in I$), as is $(b_1, b_2 + 1, b_3 - 1)$. Let us consider the possibilities for $\mathbf{b}_{N-1}$. It cannot be $\mathbf{b} + e_1 - e_2$ or $\mathbf{b} + e_1 - e_3$, as $\mathbf{b}_{N-1,1} \le \max(b_1, a_1) = b_1$. It cannot be $\mathbf{b} + e_3 - e_1$ or $\mathbf{b} + e_3 - e_2$ for a similar reason. Finally, it cannot be $\mathbf{b} + e_2 - e_1$ or $\mathbf{b} + e_2 - e_3$ as those are both in I (and in the previous paragraph we saw $\mathbf{b}_{N-1} \notin I$). Here we have a contradiction.

It remains to discuss the case $b_3 > a_3$ ($a_3 > b_3$ is symmetrical), so $a_1 \le b_1$ and $a_2 > b_2$ (so we can assume b_1 is nonzero, otherwise $\mathbf{a} = \mathbf{b}$ as there is only one boundary point with first component 0). There are two possibilities. If $(b_1 - 1, b_2 + 1, b_3) \in I$, we get a contradiction as in the above paragraph. If $(b_1 - 1, b_2 + 1, b_3) \notin I$, then $(b_1, b_2 + 1, b_3 - 1)$ is a boundary point. Moreover, because $(b_1, b_2 + 1, b_3 - 1)$ is in I by the strongly stable condition and $\{\mathbf{b}; (b_1 - 1, b_2 + 1, b_3); (b_1, b_2 + 1, b_3 - 1)\}$ is the down-graph defined by $(b_1, b_2 + 1, b_3)$, there must be a linear syzygy between $\mathbf{b}$ and $(b_1, b_2 + 1, b_3 - 1)$. Since $a_2 \ge b_2 + 1$, $a_1 \le b_1$, and $a_3 \le b_3 - 1$, $m_2(b_1, b_2 + 1, b_3 - 1) \mid m_2(\mathbf{a})$ by isotonicity. Moreover, the aforementioned linear syzygy forces $m_1(b_1, b_2 + 1, b_3 - 1) = m_1(\mathbf{b})$ and $m_3(b_1, b_2 + 1, b_3 - 1) \mid m_3(\mathbf{b})$, so $(b_1, b_2 + 1, b_3 - 1) \le \mathbf{a} \vee \mathbf{b}$ and $m(b_1, b_2 + 1, b_3 - 1) \mid \operatorname{lcm}(m(\mathbf{a}), m(\mathbf{b}))$. Moreover, $d(\mathbf{a}, (b_1, b_2 + 1, b_3 - 1)) = a_2 - (b_2 + 1) < a_2 - b_2 = d(\mathbf{a}, \mathbf{b})$, so by minimality, there is a restricted LS-path between $\mathbf{a}$ and $(b_1, b_2 + 1, b_3 - 1)$. Patching this path to the linear syzygy between $(b_1, b_2 + 1, b_3 - 1)$ and $\mathbf{b}$ gives a restricted LS-path from $\mathbf{a}$ to $\mathbf{b}$, again giving a contradiction.

In any case, we get a contradiction. Thus the proposition is proved for boundary points, and we are done. □

6. Spanning Tree Condition

Let us first describe the shape of the syzygy in the strongly stable case. We impose the following partial order on the generators of a strongly stable ideal I: we say that $\mathbf{a} \ge \mathbf{b}$ if for all k, $\sum_{i=0}^k a_i \le \sum_{i=0}^k b_i$. The strongly stable condition then guarantees that for all $\mathbf{b} \in \Delta^{\mathbb{Z}}(n, d)$ such that $\mathbf{a} \ge \mathbf{b}$, $\mathbf{b}$ must be a generator of I. Then note that here, $\mathbf{a}$ covers $\mathbf{b}$ when $\mathbf{b} = \mathbf{a} + e_i - e_j$ where $i < j$. In particular, this means that the monomials in $D(\mathbf{c}) \cap G_I$ are pairwise comparable, so that they are totally ordered.

We are interested in describing which monomials of the down-graphs are in I. To do this, take any $D(\mathbf{c}) \cap G_I$. Since we have a total order on the monomials, we can take $\mathbf{a}$ to be the largest; let $\mathbf{c} = \mathbf{a} + e_k$. Then, the set of monomials of $D(\mathbf{c})$ in I can be written as $\{\mathbf{c} - e_i | i \ge k\}$.

Theorem 6.1. *A set of isotone maps $X_1, \dots X_n$ determines a polarization of I if and only if for every $\mathbf{c} \in \Delta^{\mathbb{Z}}(n, d+1)$ such that $D(\mathbf{c}) \cap G_I \ne \emptyset$, the graph of linear syzygy edges restricted to $D(\mathbf{c}) \cap G_I$ contains a spanning tree for $D(\mathbf{c}) \cap G_I$.*

Proof. The "if" direction was done in Proposition 4.2, so we prove the "only if" direction. We will use the fact (item (β) of the Corollary to Theorem 16.3, [Mat87]) that because we always work in an

$\mathbb{N}$-graded ring (a quotient of a polynomial ring by a homogeneous ideal) and variable differences are homogeneous of degree 1, any permutation of a regular sequence is again a regular sequence. Hence, given a polarization of a strongly stable ideal I, we may choose the order of the regular sequence of variable differences by which we quotient out to recover I.

We assume that the isotone maps $\{X_i\}$ give an ideal I' which is a polarization. We shall prove that every down-graph $D(\mathbf{c})$ contains a spanning tree of linear syzygy edges for the vertices contained in G_I. For simplicity we shall assume $\mathrm{Supp}(\mathbf{c})$ has full support; the arguments work just as well in the general case. Write the set of monomials of $D(\mathbf{c})$ in I as $\{\mathbf{c}-e_i | i \geq k\}$. Then we treat this in two cases: in the first case, suppose $k < n-1$; in the second case, suppose $k = n-1$. (The case where $k = n$ is trivially true.)

Case 1. Note that if the distance between $m(\mathbf{c}-e_v)$ and $m(\mathbf{c}-e_w)$ is 2, then there is a linear syzygy between these monomials. Suppose now, for the sake of contradiction, the vertices in $D(\mathbf{c}) \cap G_I$ can be divided into distinct subsets V_1 and V_2 such that there is no linear syzygy edge between a vertex in V_1 and a vertex in V_2.

Let $m(\mathbf{c}-e_v)$ in V_1 and $m(\mathbf{c}-e_w)$ in V_2 such that the distance d between $m(\mathbf{c}-e_v)$ and $m(\mathbf{c}-e_w)$ is minimal. We must have $d \geq 3$ and the number of vertices $m \geq 3$. For simplicity we may assume $v = k$ and $w = k+1$ and that we may write

$$n(\mathbf{c}-e_{k+1}) = x_{1j_1} \cdots x_{(k-1)j_{k-1}} x_{kj_k} x_{(k+2)j_{k+2}} \cdots x_{nj_n},$$

$$n(\mathbf{c}-e_k) = x_{1i_1} \cdots x_{(k-1)i_{k-1}} x_{(k+1)i_{k+1}} x_{(k+2)i_{k+2}} \cdots x_{ni_n},$$

where $x_{pi_p} \neq x_{pj_p}$ for $p = k+2, \ldots, k+d-1$ and $x_{pi_p} = x_{pj_p}$ for $p \geq k+d$ and $p < k$.

Consider the graded ring $k[\check{X}_1, \ldots \check{X}_n]/I'$ and quotient out by the regular sequence $x_{pi_p} - x_{pj_p}$ for $p = k+3, \ldots, k+d-1$. This is a regular sequence since we began with a polarization. We get a quotient algebra $k[\check{X}'_1, \ldots \check{X}'_n]/I'$ and denote by x_p the class $\overline{x_{pi_p}} = \overline{x_{pj_p}}$ for $p \neq k, k+1, k+2$. In I' we have generators

$$\overline{m}(\mathbf{c}-e_k) = \overline{\mathbf{m}} \cdot \overline{n}(\mathbf{c}-e_k), \quad \overline{n}(\mathbf{c}-e_k) = x_1 \cdots x_{k-1} x_{(k+1)i_{k+1}} x_{(k+2)i_{k+2}} x_{k+3} \cdots x_n,$$

$$\overline{m}(\mathbf{c}-e_{k+1}) = \overline{\mathbf{m}} \cdot \overline{n}(\mathbf{c}-e_{k+1}), \quad \overline{n}(\mathbf{c}-e_{k+1}) = x_1 \cdots x_{k-1} x_{kj_k} x_{(k+2)j_{k+2}} x_{k+3} \cdots x_n.$$

Now, note that $x_{(k+2)i_{k+2})} - x_{(k+2)j_{k+2})}$ is a non-zero divisor of $k[\check{X}_1, \ldots \check{X}_n]/I$. Consider

$$(x_{(k+2)i_{k+2})} - x_{(k+2)j_{k+2})})x_1 \cdots x_{k-1} x_{kj_k} x_{(k+1)i_{k+1}} x_{k+3} \cdots x_n \cdot \overline{\mathbf{m}}.$$

It is zero in this quotient ring, and so

$$\overline{\mathbf{m}'} = x_1 \cdots x_{k-1} x_{kj_k} x_{(k+1)i_{k+1}} x_{k+3} \cdots x_n \cdot \overline{\mathbf{m}}$$

is zero in this quotient ring and so must be a generator of I' of degree $\mathbf{c}-e_{k+2}$. But then the generator of this degree in the polarization I must be

$$\mathbf{m}' = x_{1l_1} \cdots x_{(k-1)l_{k-1}} x_{kj_k} x_{(k+1)i_{k+1}} x_{(k+3)l_{k+3}} \cdots x_{nk_n} \cdot \mathbf{m}$$

where each k_p is either i_p or j_p. Hence all $l_p = i_p = j_p$ for $p < k$ and $p \geq k+d$. But then the distance between $\mathbf{m}'$ and $m(\mathbf{c}-e_k)$ is $\leq d-1$ and similarly the distance between $\mathbf{m}'$ and $m(\mathbf{c}-e_{k+1})$ is $\leq d-1$. Whether $\mathbf{m}'$ is in V_1 or in V_2, we see that this contradicts d being the minimal distance.

Case 2. Let I' be a polarization of I. In this case, assume there are only two vertices in $D(\mathbf{c}) \cap G_I$ for some $\mathbf{c}$. For the sake of simplicity, assume that $\mathbf{c}$ has full support (the argument works the same more generally) so we have that the only two vertices in this intersection are $\mathbf{c}-e_{n-1}$ and $\mathbf{c}-e_n$. For the sake of contradiction, suppose there is no edge between them. Then $X_i(\mathbf{c}-e_{n-1}) \neq X_i(\mathbf{c}-e_n)$ for some $i < n-1, n$. Let $x_{i,a}$ be a variable such that it divides $m_i(\mathbf{c} - e_{n-1})$ but not $m_i(\mathbf{c} - e_n)$, and let $x_{i,b}$ be a variable dividing $m_i(\mathbf{c} - e_n)$ but not $m_i(\mathbf{c} - e_{n-1})$.

Consider the graded ring $\tilde{S}/I'$ where $\tilde{S} = k[\check{X}_1, \ldots, \check{X}_n]$ and $\check{X}_j := \{x_{j,1}, \ldots, x_{j,d_j}\}$. Quotient out by the regular sequence $x_{p,1} - x_{p,j}$ for all $p \neq i$ and $2 \leq j \leq d_p$, and if $d_i \geq 3$, then quotient again by the regular sequence $x_{i,k} - x_{i,j}$ for some fixed $k \neq a, b$ and all $j \in [d_i] \setminus \{a, b, k\}$. We know that the union of these two sequences of variable differences is a regular sequence because we assume we started with a polarization, and we are also allowed to choose the order of the regular sequence of variable differences by which we quotient out.

The result is that we are left with a quotient algebra R/J where $R = k[x_1, \ldots, x_{i-1}, x_i, x_{i,a}, x_{i,b}, x_{i+1}, \ldots, x_n]$ and J has generators which we denote $\overline{m}(\mathbf{a})$ for all exponent vectors $\mathbf{a}$ in the minimal generating set of I (if $d_i = 2$ then there is no x_i variable, but this does not change the argument). Now, we have that

$$(x_{i,a} - x_{i,b}) \cdot \mathbf{x}^{\mathbf{c}-e_i} = 0$$

because each term divides one of $\overline{m}(\mathbf{c} - e_n)$ or $\overline{m}(\mathbf{c} - e_{n-1})$, but any monomial with exponent vector $\mathbf{c} - e_i$ where $i < n-1$ is not in I by assumption, so $(x_{i,a} - x_{i,b})$ is a zero-divisor, yielding a contradiction.

$\square$

7. Alexander Duals and Associated Primes of Polarizations

The aim of this section is to better understand the Alexander duals and associated primes of polarizations. We prove theorem 7.7, which gives us the form of Alexander duals of polarizations of any monomial ideal.

We begin this section with some background definitions and discussion of results in the literature before presenting our work.

Definition 7.1 (Associated prime). Let R be a Noetherian ring and M a finitely generated R-module. A prime ideal $P \subset R$ is an *associated prime ideal* of M, if there exists an element $x \in M$ such that $P = \operatorname{ann}(x)$, where $\operatorname{ann}(x)$ is the annihilator of x, that is to say, $\operatorname{ann}(x) = \{a \in R : ax = 0\}$. The set of associated prime ideals of M is denoted $\operatorname{Ass}(M)$.

In our setting we know much about what these associated primes look like. In particular, for $I \subset R = k[x_1, \ldots, x_n]$ a monomial ideal, $\operatorname{Ass}(R/I)$ is a finite set of prime ideals generated by monomials. Further, for I square-free, I can be written as an intersection of its associated primes, and these associated primes are all generated by variables (1.3.6 in [HH11]).

Definition 7.2 (Alexander Dual). Let I be a square-free monomial ideal in a polynomial ring S . The Alexander dual ideal $I^\vee$ of I is the monomial ideal in S whose monomials are precisely those that have nontrivial common divisor with every monomial in I. Equivalently, they have a nontrivial common divisor with every generator of I

Remark 7.3. Given a square-free monomial ideal I, an important relationship exists between its Alexander dual $I^\vee$ and its associated primes $\operatorname{Ass}(I)$. In particular, the minimal generators of the Alexander dual of I correspond to the variables that generate the associated primes of I. For example, if $I^\vee = (x_1x_3x_4, x_2x_3x_5)$, then (x_1, x_3, x_4) and (x_2, x_3, x_5) are associated primes of I.

This relationship motivates us to want to know what form the generators of our Alexander duals take. We introduce a helpful bit of terminology from [AFL22].

Definition 7.4 (Color Classes, Rainbow Monomials)**.** We call the set of all variables sharing their first index $\{x_{i,1},\dots,x_{i,m}\}$ the i–color class. We call a monomial in degree d a *rainbow* monomial when it is of the form $x_{1,j_1}\dots x_{d,j_d}$, a product of exactly one variable from each color class.

Notably, in [AFL22], the authors prove that the class of degree d rainbow monomials with linear resolution are exactly the class of ideals Alexander dual to polarizations of Artinian monomial ideals.

Proposition 7.5. *The class of ideals generated by rainbow monomials and with n-linear resolution is precisely the class which is Alexander dual to the class of polarizations of Artinian monomial ideals in n variables. More precisely:*

a. *Let J be a polarization of an Artinian monomial ideal I in $k[x_1,\dots,x_n]$. The Alexander dual ideal of J is generated by rainbow monomials and has n-linear resolution.*
b. *If an ideal J' is generated by rainbow monomials and has n-linear resolution (and every variable in the ambient ring occurs in some generator of the ideal), then its Alexander dual J is a polarization of an Artinian monomial ideal in n variables.*

In pursuit of a similar result for monomial ideals in general, we introduce the following definition.

Definition 7.6 (Weakly-Rainbow)**.** We say a monomial is *weakly-rainbow* if it is generated by at most one variable from each color class.

We now present a result in the direction of generalizing Proposition 7.5.

Theorem 7.7. *If $I \subset S = k[x_{1,1},\dots,x_{1,d},\dots,x_{n,1},\dots,x_{n,d}]$ is a polarization of any monomial ideal $J \subset k[x_1,\dots,x_n]$, then the generators of $I^\vee$ are weakly-rainbow.*

We will need two technical lemmas to prove this result.

Lemma 7.8. *Let $S = k[x_{1,1},\dots,x_{1,d},\dots,x_{n,1},\dots,x_{n,d}]$, and let $I \subset S$ be a monomial ideal whose generators have degree strictly greater than 1. Let $x_{i,s}-x_{i,k}$ be a non-zerodivisor on $R = S/I$, and $x_{i,j}-x_{i,l} \neq 0$ a zerodivisor on R for some i,j,k,l. Then $x_{i,j}-x_{i,l}$ is non-zero and a zerodivisor on $R' = R/(x_{i,s}-x_{i,k})$.*

Proof. First we check that if $x_{i,j}-x_{i,l}$ is nonzero in R, then it must also be nonzero in R'. This follows from the following string of implications:

$$\begin{aligned}
&x_{i,j}-x_{i,l} = 0 \text{ in } R' \\
\implies &x_{i,j}-x_{i,l} \in (x_{i,s}-x_{i,k})I \\
\implies &x_{i,j}-x_{i,l}+I = (v+I)*(x_{i,s}-x_{i,k})+I \text{ for some } v \in S \\
\implies &x_{i,j}-x_{i,l} = v*(x_{i,s}-x_{i,k})+r \text{ for some } v \in S, \text{ and } r \in IS.
\end{aligned}$$

By assumption I is a monomial ideal in degree strictly greater than 1, so r in the last line above contributes no degree 1 term. Hence the degree 1 terms of $x_{i,j}-x_{i,l}$ must equal the degree 1 terms of $v*(x_{i,s}-x_{i,k})$. But $x_{i,s}-x_{i,k}$ already is composed of degree 1 terms, hence we must have that $x_{i,j}-x_{i,l} = t*(x_{i,s}-x_{i,k})$ for some unit t. Then in R we have that $x_{i,j}-x_{i,l}$ a zero divisor implies that $x_{i,s}-x_{i,k}$ is a zero divisor, a contradiction. Hence $x_{i,j}-x_{i,l} \neq 0$ in R'.

Now suppose, seeking contradiction, that $x_{i,j}-x_{i,l}$ is a non-zero divisor in R'. Our goal is to show that this implies that it must have also been a zero divisor in R. Recall that since $x_{i,j}-x_{i,l} \neq 0$ is a zero divisor in R, there exists an $0 \neq m \in R$ such that $(x_{i,j} - x_{i,l})m = 0 \in R$. Then since $(x_{i,j} - x_{i,l})m = 0 \in R$, $(x_{i,j} - x_{i,l})m = 0 \in R'$. Hence for $x_{i,j} - x_{i,l} \neq 0$ to be a non-zero divisor in R', we need that $m = 0$ in R'.

First we check that, for any element m such that m is nonzero in R but equal to 0 in R', we have that $b = x_{i,s} - x_{i,k}$ must divide m. Take m in R and suppose, seeking contradiction, that $\sup(r | m = (b)^r * m_r) = \infty$ for some $m_r \in R$). Then given any $r \geq 0$, we have that $m_0 = m = b^r * m_r$. Suppose that $m_t \neq b * m_t$ for all t. Then we can construct an infinite chain of ideals $(m_0) = (b * m_1) \subsetneq (m_1) \subsetneq (m_2) \subsetneq \dots$, but this contradicts that S/I is Noetherian. Thus there exists some r such that $m_r = b * m_r$ in R.

Then $(b-1) * (m_r) = 0$ in R. Take $q = (x_{1,1}, \dots, x_{1,d}, \dots, x_{n,1}, \dots, x_{n,d})$ to be the graded maximal ideal in S. Recall that $m_r = b * m_r$ in R implies that $m_r = b * m_r + u$ in S for some $u \in I$. Since b has no degree 0 components and I has no degree 0 components, then m_r has no degree zero components. Hence we have that $b, m_r \in q$ and $I \subset q$.

Therefore, we have that since $m_r \neq 0$ in R, it is also nonzero in the localization R_q. To see this, suppose it is not. Then there exists some element in R outside of q that multiplies m_r to zero. But since this element is outside of q, it must have a degree zero component. Hence we get that $u * m_r + v * m_r \in I$ for some unit u and an element $v \in S$ where v has no degree zero components (or is zero). Take s_i the non-zero monomial of m_r in R with minimal degree; since s_i is non-zero in R, it is not an element in I. Then since $v * m_r$ has components with degrees strictly greater than 0 (or else is zero), $u * s_i$ is a monomial of minimal degree in $u * m_r + v * m_r \in I$. But a polynomial f belongs to I if and only if all monomials in f appearing with a nonzero coefficient belong to I, hence we conclude that $u * s_i \in I$. But then $s_i \in I$, contradicting our original choice of s_i.

Then notice also that $b \in q$ implies that $(b-1)$ is not in q, and hence is a unit in the localization R_q. But then since $(b-1) * m_r = 0$ in S_{1_q}, we get that $b-1$ is both a unit and a zero divisor in S_{1_q}, a contradiction. Hence there exists an $r = max(r | m = b^r * m_r)$. Hence $m_r \neq 0$ in R'.

Now we have two cases:

Case 1: suppose that $m_r * (x_{i,j} - x_{i,l}) = 0$ in R. Then $m_r * (x_{i,j} - x_{i,l}) = 0$ in R', contradicting that $x_{i,j} - x_{i,l}$ is a non-zero divisor in R'.

Case 2: suppose that $m_r * (x_{i,j} - x_{i,l}) \neq 0$ in R. Then $b * (x_{i,j} - x_{i,l}) * m_r = 0$, yet $(x_{i,j} - x_{i,l}) * m_r \neq 0$, contradicting that b is a non-zero divisor on R.

Hence we conclude that $x_{i,j} - x_{i,l} \neq 0$ is a zero divisor in R.

□

With this lemma, we now only need the following.

Lemma 7.9. *Let $S = k[x_{1,1}, \dots, x_{1,d}, \dots, x_{n,1}, \dots, x_{n,d}]$. If $I \subset S$ is a polarization of some monomial ideal J, I separated in each x_i variable by some regular sequence $r_{i,1}, \dots, r_{i,d_i-1}$ of variable differences, and J generated by generators with degrees strictly greater than 1, then $x_{i,j} - x_{i,l}$ is a non-zero divisor in $R = S/I$.*

Proof. Suppose that $x_{i,j} - x_{i,l} \neq 0$ is a zero divisor on $R = S/I$ for some i, j, k. Let $r_{i,d_i-1} = x_{i,s} - x_{i,l}$ for some be the first element of the regular sequence $r_{i,1}, \dots, r_{i,d_i-1}$ of variable differences separating J to I. Then notice that we are in the setting of Lemma 7.8. Applying the lemma, we obtain that $x_{i,j} - x_{i,l} \neq 0$ is a zero divisor in $R' = R/(x_{i,s} - x_{i,k})$. But notice that $R' \cong S/(I + (x_{i,s} - x_{i,k})) \cong \tilde{S}/I'$ for $\tilde{S}$ the polynomial ring obtained by replacing each $x_{i,s}$ by $x_{i,k}$ and I' a monomial ideal with the degrees of its generators the degrees of the generators of I obtained by replacing each $x_{i,s}$ in I by $x_{i,k}$. Hence the necessary conditions are satisfied to apply Lemma 7.8 again to find that $x_{i,j} - x_{i,l} \neq 0$ is a zero divisor in $R'' = R'/(r_{i,d_i-2})$. Continuing in this manner, by quotienting out by our regular sequence we find that $x_{i,j} - x_{i,l} \neq 0$ is a non-zero zerodivisor in $R/(r_{i,1},...,r_{i,d_i})$. But by quotienting out our regular sequence, we have collapsed all the separations in the x_i variable, hence $x_{i,j} = x_{i,l}$ in $R/(r_{i,1},...,r_{i,d_i})$, contradicting that $x_{i,j} - x_{i,l} \neq 0$ in this quotient.

□

With these two lemmas, we are now able to prove our theorem.

Proof of Theorem 7.7. Let $I \subset S = k[x_{1,1},\ldots,x_{1,d},\ldots,x_{n,1},\ldots,x_{n,d}]$ be a polarization of a monomial ideal $J \subset \mathfrak{m}^2$. Suppose that the generators of $I^\vee$ are not weakly-rainbow. Then there exists some generator g of $I^\vee$ that contains distinct two variables from the same color class, $x_{i,j}$ divides g and $x_{i,k}$ divides g with $j \neq k$ (by remark 7.3 $j \neq k$). Then by remark 7.3, $x_{i,j}, x_{i,k}$ are both contained in the same associated prime P of R/I, hence $x_{i,j} - x_{i,k} \in P$ is a zerodivisor on $R = S/I$. However, since I is a polarization of a monomial ideal J with no generators of degree less than 2, by lemma 7.9 $x_{i,j} - x_{i,k}$ is a non-zerodivisor on S/I for all $j \neq k$. Hence we have a contradiction, and so g must be weakly-rainbow.

If $I \subset S = k[x_{1,1},\ldots,x_{n,1}]$ is a polarization of a monomial ideal J with some degree 1 generators, the above proof follows, noting that the degree 1 generators of J are each members of a color class containing only one variable.

Hence we conclude the generators of $I^\vee$ are weakly-rainbow.

□

8. Stanley-Reisner complexes and shellability

In this section, we consider the shellabity of the Stanley-Reisner complexes of polarizations of the power of a maximal ideal. We first recall key definitions and lemmas, and then we present our strategy to produce a shelling order for the Stanley-Reisner complexes associated to any polarization of $I = (x_1,\ldots,x_n)^d$.

Notation 8.1. Let I be an Artinian monomial ideal in $S = k[x_1,\ldots,x_n]$, for each variable i there is a minimal generator of I of the form $x_i^{d_i}$. Let $\check{X}_i = \{x_{i,1},\ldots,x_{i,d_i}\}$ be a set of double-indexed variables of color i, and let $X = \{x_{11},\ldots,x_{nd_n}\}$ be the union of all these variables. Denote a polarization of I in $\tilde{S} = k[X]$ by $\tilde{I}$. As an abuse of notation, we will also often let X denote the product of all the variables in X. If a squarefree monomial $\mathbf{m}$ is a product of a subset of variables in X, we will say that $\mathbf{m} \subset X$. For any squarefree monomial ideal J, denote its Alexander dual by $J^\vee$.

Definition 8.2. For a squarefree monomial ideal I, the *Stanley-Reisner complex* of I is the simplicial complex consisting of the monomials not in I,

$$\Delta_I = \{\mathbf{m} \subset X | \mathbf{m} \notin I\}.$$

Remark 8.3. For any squarefree monomial ideal I, the facets of $\Delta_{\tilde{I}}$ are of the form $\frac{X}{\mathbf{m}}$, where $\mathbf{m}$ is a monomial generator of $\tilde{I}^\vee$. Hence in this section, we talk about an ordering of the facets of $\Delta_{\tilde{I}}$ and an ordering of the generators of $\tilde{I}^\vee$ synonymously.

Definition 8.4. An ordering $F_1, \dots F_t$ of the facets of a simplicial complex Δ is a *shelling* if, for each j with $1 < j \le t$, the intersection

$$\left(\bigcup_{i=1}^{j-1} F_i \right) \cap F_j$$

is a nonempty union of facets of ∂F_j. If there exists a shelling of Δ, then Δ is called *shellable*.

The following rephrasings of the condition for shellability will be useful for our purposes.

Lemma 8.5. *For a facet F_i of a simplicial complex I, denote by F_i^c its complement $\frac{X}{F_i}$. The following are equivalent:*

(1) An order $F_1, \dots, F_t$ of the facets of a simplicial complex Δ is a shelling.

(2) For every i and k with $1 \le i < k \le t$, there is some j with $1 \le j < k$ and an $x \in F_k$ such that $F_i \cap F_k \subseteq F_j \cap F_k = F_k - \{x\}$.

(3) The ordering $F_1^c, \dots, F_t^c$ of the generators of $I^\vee$ is a linear quotient ordering, that is, for any $1 \le k \le t$ the colon ideal $(F_1^c, \dots, F_k^c) : (F_{k+1}^c)$ is generated by a subset of the variables of X.

We also have the following lemma and conjecture from [AFL22].

Lemma 8.6. [AFL22, Lemma 3.1] *Let $\Delta_{\tilde{I}}$ be the simplicial complex associated to the polarization $\tilde{I}$ of an Artinian monomial ideal I. Then every codimension one face of is contained in one or two facets. If I is not a complete intersection, then at least once there is a codimension one face contained in exactly one facet.*

By work of Danaraj and Klee [DK74], any shellable simplicial complex with the property in Lemma 8.6 is a simplicial ball or sphere, leading the authors of [AFL22] to conjecture the following.

Conjecture 8.7. [AFL22, Conjecture 3.2] *The simplicial complex $\Delta_{\tilde{I}}$ associated to a polarization $\tilde{I}$ of an Artinian monomial ideal I, is a simplicial ball, save for the case when I is a complete intersection, when it is a simplicial sphere.*

Note that to prove Conjecture 8.7 it suffices to show that any polarization of an Artinian monomial ideal is shellable. In [AFL22], this was proven in for ideals of the form $(x_1, \dots, x_n)^d$ when $n = 3$. We aim to generalize this result to arbitrary n. To do this, we first present the following theorem, appearing in [AFL22], which is a reformulation of the original statement in [Nem21], which itself was a rephrasing of a result in [FGM18]. This theorem gives a complete characterization of rainbow monomial ideals with linear resolution, and it is a key tool for our strategy.

Theorem 8.8. *Let I be generated by rainbow monomials in d colors. Then I has a d-linear resolution if and only if both of the following two conditions hold:*

(a) Whenever m_1 and m_2 are two rainbow monomials in I (i.e. generators of degree d) with $lcm(m_1, m_2)$ of degree $\ge d + 2$, there is a third distinct rainbow monomial m_3 in I dividing this least common multiple.

(b) Whenever m_1 and m_2 are two rainbow monomials not in I with $lcm(m_1, m_2)$ of degree $\ge d + 2$, there is a third distinct rainbow monomial m_3 not in I dividing this least common multiple.

By Proposition 7.5, the theorem above gives a complete characterization of Alexander duals of polarizations of Artinian monomial ideals. In particular, part (a) of the Theorem 8.8 implies that for any two monomial generators m_1, m_2 of $\tilde{I}^\vee$ where I is an Artinian monomial ideal, there is a sequence of monomial generators $m_1 = p_1, p_2, \ldots, p_t = m_2$ such that there is a linear relation between p_i and p_{i+1}, and each p_i divides $\operatorname{lcm}(m_1, m_2)$: we can see this inductively – in the base case, when the distance between two monomial generators is 1, this is automoatically true, and we canse use part (a) of the above theorem for the inductive step to strictly decrease the distance.

The following tool will be central to constructing our shelling order.

Definition 8.9 (Facet-ridge graph)**.** Given a d-dimensional pure simplicial complex Δ, any $(d-1)$-dimensional face of it is called a *ridge*. The *facet-ridge graph* G_Δ of a pure simplicial complex Δ is the graph whose vertices are facets of Δ, and two facets are connected by an edge if they share a common ridge.

Remark 8.10. Notice that one can equivalently view a facet-ridge graph G_Δ as the graph of linear syzygies of the Alexander dual $G_{I^\vee}$ by viewing each vertex labeled by a facet F_i as instead being labeled by the generator F_i^c of the Alexander dual and each edge corresponding to a linear syzygy between two generator of $I^\vee$. More precisely, there is an edge between two vertices corresponding to the minimal generators $\mathbf{a}$ and $\mathbf{b}$ of $I^\vee$ if $\mathbf{b} = \frac{x_{i,j}}{x_{i,k}}\mathbf{a}$ for some j, k.

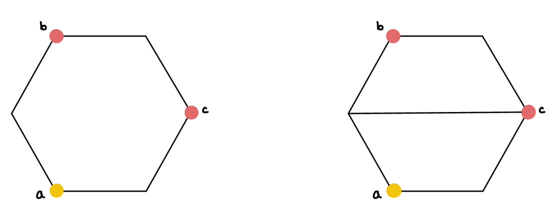
Definition 8.11. Given two minimal generators $\mathbf{a}, \mathbf{b}$ of the Alexander dual $J = \tilde{I}^\vee$, define their *distance* $d(\mathbf{a}, \mathbf{b})$ to be the length of the shortest path between them in the graph G_J. Equivalently, it is $n - \deg(\gcd(\mathbf{a}, \mathbf{b}))$, since there exists a linear syzygy path in G between m_1 and m_2 of length exactly that.

Now we will introduce a notion of well-connectedness in graphs, and show that G being well-connected is a sufficient for the shellability of $\Delta_{\tilde{I}}$. Then we will show our reasons to suspect that G is well-connected.

Definition 8.12 (Well-connected)**.** A graph is *well-connected* if for any vertices a, b, c, there exists a shortest path from b to c such that the distance from a to anything on the path is $\leq \max(d(a,b), d(a,c))$.

Notice that by repeatedly applying Definition 8.12, we actually have that if a graph G is well-connected, for any vertices a, b, c in G, there exists such a shortest path that is monotonic in its distance to a.

Example 8.13. The hexagon graph is not well-connected since $d(a,b) = 2, d(a,c) = 2, d(b,c) = 2$, but the only length 2 path from b to c goes through a point that is distance 3 from a. However, the hexagon graph modified by connecting a pair of antipodal points by an edge is well-connected.



This example suggests that a good heuristic for well-connectedness in G is having enough relations between the generators of $\tilde{I}^\vee$.

Question 8.14. *Let $\tilde{I}^\vee$ be a polarization of an Artinian monomial ideal and let Δ be its associated Stanley-Reisner complex. Is G_Δ well-connected?*

We motivate this question with an example.

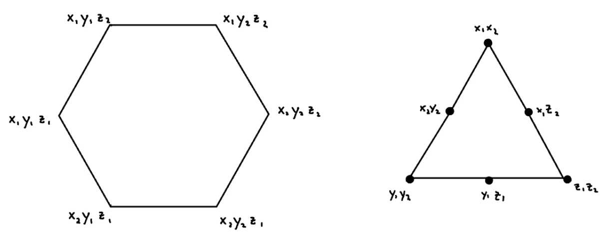
Example 8.15. We show an example of a preseparation of $I = (x, y, z)^2$ that leads to G being the hexagon graph, and show that it is not a polarization. Consider the preseparation

$$J = (x_1x_2, x_2y_2, x_1z_1, y_1y_2, y_1z_1, z_1z_2).$$

This gives the Alexander dual

$$J^\vee = (x_1y_1z_1, x_1y_1z_2, x_1y_2z_2, x_2y_2z_2, x_2y_2z_1, x_2y_1z_1),$$

which has the hexagon graph as its linear syzygy graph. However, notice that in the linear syzygy graph of J, the down-triangle does not have a spanning tree – in fact there is no relations in the down-triangle. Hence in this sense, J is far from a valid polarization of I – it has too few relations.



This suggests that preseparations that causes the hexagonal situation don't have enough relations to be a polarization. This also motivates the following observation about relations in the Alexander dual. In particular, construct the dual of the linear syzygy graph of I by having a vertex for each up-graph and an edge between two vertices if their up-graphs share a vertex and there is at least one generator in the Alexander dual related to both up-graphs. The following is an observation about this dual graph.

Proposition 8.16. *Suppose $d = 3$. Then for each edge in the down-graph $D(\mathbf{c})$, there is a parallel edge in the associated up-graph in the dual graph.*

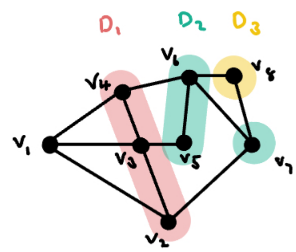
Proof. We consider the edge between the vertices $\mathbf{c} - e_2$ and $\mathbf{c} - e_3$; the other cases are similar. Note that since there is an edge, $\mathbf{c} - e_2$ and $\mathbf{c} - e_3$ must have the same x-component. For simplicity, suppose x_1 is a factor in both. This also means that the vertex $\mathbf{c} - e_1$ is nonzero in the y and z components; for simplicity, suppose y_1z_1 is a factor. Hence $\mathbf{c} - e_1 - e_3 + e_2$ has a factor of y_1 and $\mathbf{c} - e_1 - e_2 + e_3$ has a factor of z_1. Then $x_1y_1z_1$ is a generator of the Alexander dual related to both up-triangles $U(\mathbf{c} - e_1 - e_3)$ and $U(\mathbf{c} - e_1 - e_2)$, so there is an edge between their associated vertices in the dual graph. $\square$

The following algorithm gives an order on the generators on the Alexander dual. This is a variant of breadth-first search. We show that if Δ is the Stanley-Reisner complex of a polarization of an Artinian monomial ideal, running this algorithm on a well-connected G_Δ gives a shelling order for Δ.

Algorithm 8.17. *Let C be a graph. For each connected component of C, add an arbitrary vertex v not already in the order, if any, to the end of the order. When adding each v, let the set of vertices in C distance i from v that have not already been added to the order be D_i. Recurse on each of the subgraphs induced by D_i, with i in increasing order.*

Note that this algorithm gives an order on the whole vertex set when we run it on G since G is connected and we only add a vertex if it has not appeared, so each vertex appears only once. It terminates since each D_i is of a strictly smaller size than the C it was derived from.

Example 8.18. Here is an example of a possible order generated by Algorithm 8.17 on a graph. We start at v_1 and mark the sets D_i at distance i from v_1. Note that the induced subgraph on each D_i is connected, we simply choose our ordering by simple BFS on each D_i, successively.



Lemma 8.19. *Let I be a rainbow monomial ideal with linear resolution. Let $v_1, \dots, v_g$ be an ordering of the generators of I generated by Algorithm 8.17. Then for each v_i, v_j where $i < j$, there is a shortest path from v_i to v_j such that each vertex on the path comes before v_j.*

Proof. Note here that in our setting, the shortest path between two variables is the number of variable differences by the construction of the graph of linear syzygies. From the algorithm, we know that $d(v_1, v_i) \le d(v_1, v_j)$. From well-connectedness of G, we get that there is a monotonic path $v_i = p_0, \dots, p_m = v_j$ where $d(v_1, p_k) \le d(v_1, v_j)$. If $d(v_1, p_k) < d(v_1, v_j)$, p_k necessarily comes before v_j since the smaller distance vertices are added to the ordering first. If $d(v_1, p_k) = d(v_1, v_j)$, we know that p_k and v_j are in the same connected component, by the monotonicity of the path. For two monomials to differ from v_1 both by the same number of variables and to have a linear relation between them, the variables where they differ from v_1 must be the same and these are the same between the two other than one. This means that if we have a shortest path, k is necessarily either 0 or m, so p_k is in the ordering. Hence, for each v_i, v_j where $i < j$, there is a monotonic shortest path from v_i to v_j such that each vertex on the path comes before v_j. $\square$

Theorem 8.20. *Let $\tilde{I}$ be a polarization of $I = (x_1, \dots, x_n)^d$. If the linear syzygy graph G on the Alexander dual is well-connected, the order $v_1, \dots, v_t$ given by Algorithm 8.17 is a shelling order of $\Delta_{\tilde{I}}$. In other words, $\Delta_{\tilde{I}}$ is shellable.*

Proof. We can check that this gives a shelling: F_j corresponding to v_j has a nonzero intersection with $\bigcup_{i=1}^{j-1} F_i$ since v_j is connected to the subgraph on $v_1, \dots, v_{j-1}$ since it is obtained from breadth-first search, so there is a vertex distance 1 from it appearing before it, which corresponds to a facet sharing a boundary face with F_j. Further, the intersection is a union of facets of ∂F_j since for any F_i where $i < j$, the intersection $F_i \cap F_j$ is contained in the facet corresponding to p_{m-1}, which shares precisely one facet of ∂F_j with F_j. Hence, this gives a shelling, and $\Delta_{\tilde{I}}$ is shellable. $\square$

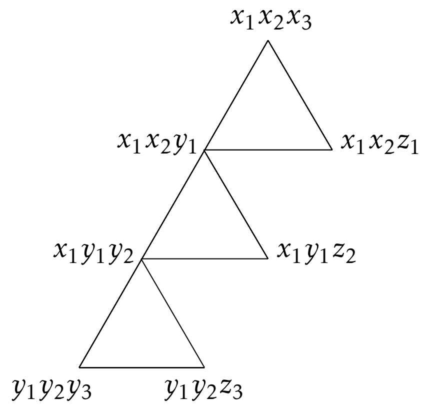


FIGURE 6. The pyramidal polarization $J_{3,3}$.

9. A CLASS OF RIGID POLARIZATIONS OF STRONGLY STABLE IDEALS

In [Loh13], Lohne shows that points on the associated Hilbert scheme $H^{p(z)}_{\mathbb{P}^{n-1}}$ corresponding to the standard and box polarizations of an ideal of the form $(x_1,\ldots,x_n)^d \subset k[x_1,\ldots,x_n]$ are smooth. We ask if this is true for (those specific) polarizations of strongly stable ideals.

One idea that we can consider is determining when such polarizations I are *rigid*; that is, their second cotangent cohomology modules $T^2_{S/I}$ are 0 (see Section 3 of [Har10] for details about the T^2 functor). While this is a sufficient condition for smoothness of the corresponding point [FGI+05, Corollary 6.2.5], it is typically too strong. For instance, a computation with Macaulay2 shows that T^2 for both the standard and box polarizations of the ideal $(x,y,z)^2$ are nonzero (they are 3- and 1-dimensional, respectively), even though they correspond to smooth points by work of Lohne; indeed, out of the many examples we've computed, T^2 is rarely 0. However, we may define a specific polarization of a certain class of strongly stable ideals that indeed have vanishing T^2:

Definition 9.1. For each $n,d \geq 1$, define $I_{n,d}$ to be the ideal in n variables generated by the monomials corresponding to up-triangles $U(\mathbf{c}_i)$, where $1 \leq i \leq d$ and $\mathbf{c}_i = (d-i, i-1, 0, \ldots, 0)$.

For instance, when $n = d = 3$, $I_{n,d}$ is generated by $\{x^3, x^2y, x^2z, xy^2, xyz, y^3, y^2z\}$. It is easy to see that $I_{n,d}$ is always strongly stable; in fact, it is the strongly stable closure of the ideal generated by $x_2^{d-1}x_n$.

Definition 9.2. Define the *pyramidal polarization* $J_{n,d}$ of $I_{n,d}$ as follows: the generator of $J_{n,d}$ corresponding to $(d-i+\epsilon_1, i-1+\epsilon_2, \epsilon_3, \ldots, \epsilon_n) \in U(\mathbf{c}_i)$ is $x_{1,1}\ldots x_{1,d-i+1}x_{2,1}\ldots x_{2,i-1}$ if $\epsilon_1 = 1$, $x_{1,1}\ldots x_{1,d-i}x_{2,1}\ldots x_{2,i}$ if $\epsilon_2 = 1$, and $x_{1,1}\ldots x_{1,d-i}x_{2,1}\ldots x_{2,i-1}x_{k,i}$ if $\epsilon_k = 1$ for $k \geq 3$.

For instance, the graph of linear syzygies of $J_{3,3}$ is depicted above in Figure 6, and the linear syzygy graph suggests the origin of the name "pyramidal polarization". Using the down-graph criterion (Proposition 4.2) it is easily verified that $J_{n,d}$ is in general a polarization of $I_{n,d}$.

Theorem 9.3. $T^2_{J_{3,d}} = 0$. *In particular, $J_{3,d}$ determines a smooth point $x_{3,d}$ on its associated Hilbert scheme.*

The proof of this theorem is mostly computational (which can be generalized to $J_{n,d}$), so we break it up into smaller steps. First, we set up the notation and prove some useful lemmas.

Let $S = k[x_1,\dots,x_d,y_1,\dots,y_d,z_1,\dots,z_d]$ be the polynomial ring containing $J_{3,d}$. Using the notation of [Har10], choose $F = S^{2d+1}$ generated by $e_1,\dots,e_{2d+1}$, so there is a surjection $F \twoheadrightarrow J := J_{3,d}$ sending e_i to $x_1 \dots x_{d-\frac{i+1}{2}+1} y_1 \dots y_{\frac{i+1}{2}-1}$ if i is odd, and $x_1 \dots x_{d-\frac{i}{2}} y_1 \dots y_{\frac{i}{2}-1} z_{\frac{i}{2}}$ if i is even. For instance, in the above $J_{3,3}$ example, e_1 is sent to $x_1x_2x_3$, e_2 is sent to $x_1x_2z_1$, e_3 is sent to $x_1x_2y_1$, and so on in a zigzag pattern until $e_7 \mapsto y_1y_2y_3$.

Let Q be the kernel of this surjection, so Q is generated by the $3d$ linear syzygies between generators of J. In particular, Q is generated by the

$$z_ie_{2i-1} - x_{d-i+1}e_{2i}, \quad y_ie_{2i-1} - x_{d-i+1}e_{2i+1}, \quad y_ie_{2i} - z_ie_{2i+1},$$

ranging over $1 \le i \le d$. When we refer to "generators of Q" (or its quotient module Q/F_0, defined in the next paragraph), we will always mean these generators.

Let $F_0 \subseteq Q$ be the submodule generated by the *Koszul relations* between the generators of $J_{3,d}$, that is, relations of the form $j_re_s - j_se_r$, where j_r is the image of e_r in $J_{3,d}$. Then with the natural S/J-module map $\theta : Q/F_0 \to F \otimes_S S/J \cong F/JF \cong (S/J)^{2d+1}$, we wish to show that $\mathrm{Hom}_{S/J}(F/JF, S/J) \to \mathrm{Hom}_{S/J}(Q/F_0, S/J)$ is surjective. To this end, given a map $\varphi \in \mathrm{Hom}_{S/I}(Q/F_0, S/I)$, we wish to construct $b_1,\dots,b_{2d+1}$ such that the map $\psi : (S/J)^{2d+1} \to S/J$ sending e_i to b_i induces φ. We will eventually construct the b_i's inductively.

Write

$$\varphi(z_ie_{2i-1} - x_{d-i+1}e_{2i}) =: a_{3i-2}, \quad \varphi(y_ie_{2i-1} - x_{d-i+1}e_{2i+1}) =: a_{3i-1}, \quad \varphi(y_ie_{2i} - z_ie_{2i+1}) =: a_{3i}.$$

Lemma 9.4. *(1) For all $1 \le i \le d$, every generator of J is divisible by x_{d-i+1}, y_i, or z_i, but there are no generators of J divisible by $x_{d-i+1}y_i$ or $x_{d-i+1}z_i$.*

(2) Every single generator of J is divisible by either x_{d-i} or y_i.

Proof. Clear from the construction. □

Lemma 9.5. *For all i, $-y_ia_{3i-2} + z_ia_{3i-1} = x_{d-i+1}a_{3i}$. In particular, we may express the a_k's in the following form:*

$$a_{3i-2} = x_{d-i+1}f_{3i-2} + z_ih_{3i-2}, \quad a_{3i-1} = x_{d-i+1}f_{3i-1} + y_ig_{3i-1}, \quad a_{3i} = y_ig_{3i} + z_ih_{3i},$$

for $f_k, g_k, h_k \in S/I$.

Proof. The first statement is immediate. For the second statement, note that $x_1 \dots x_{d-i} y_1 \dots y_{i-1}$ kills each of a_{3i-2}, a_{3i-1} and a_{3i}. Because every generator of J is either divisible by x_{d-i+1}, y_i, or z_i (Lemma 9.4), and $x_1 \dots x_{d-i} y_1 \dots y_{i-1}$ is divisible by none of those, it follows from Proposition 1.2.2 in [HH11] that $J : (x_1 \dots x_{d-i} y_1 \dots y_{i-1}) \subseteq (x_{d-i+1}, y_i, z_i)$. Each of the a_k's in question is in the colon ideal, so they may be written in the form $x_{d-i+1}f_k + y_ig_k + z_ih_k$ for $f_k, g_k, h_k \in S/I$.

Hence

$$-y_i(x_{d-i+1}f_{3i-2} + y_ig_{3i-2} + z_ih_{3i-2}) + z_i(x_{d-i+1}f_{3i-1} + y_ig_{3i-1} + z_ih_{3i-1}) = x_{d-i+1}(x_{d-i+1}f_{3i} + y_ig_{3i} + z_ih_{3i}).$$

From this, we see, for instance, that $y_i^2 g_{3i-2} \in (z_i, x_{d-i+1})$. Lifting up to S, we know that $(z_i, x_{d-i+1}) + J$ is a square-free monomial ideal, hence radical, so $y_ig_{3i-2} \in (z_1, x_d)$. Similarly, we see that $z_ih_{3i-1} \in (y_i, x_{d-i+1})$ and $x_{d-i+1}f_{3i} \in (y_i, z_i)$. Hence we may rewrite $a_{3i-2} = x_{d-i+1}f_{3i-2} + z_ih_{3i-2}$, $a_{3i-1} = x_{d-i+1}f_{3i-1} + y_ig_{3i-1}$, and $a_{3i} = y_ig_{3i} + z_ih_{3i}$. □

We next describe relations between generators of Q/F_0 that form a "chain". These chains correspond to LS-paths in the graph of linear syzygies of J, and to each chain of linear syzygies we may associate a vanishing linear combination.

Definition 9.6. Let $q_1, \dots, q_k$ be a sequence of generators of Q/F_0 with $k \geq 2$. We may write each q_i as $c_i e_{r_i} - d_i e_{s_i}$, where c_i, d_i are monomials of degree 1 and r_i, s_i are indices between 1 and $2d-1$. We call such a sequence an *LS-chain* if:

(1) $s_i = r_{i+1}$ for each $1 \leq i \leq k-1$,

(2) $r_1 < r_2 < \ldots < r_k < s_k$, and

(3) $r_2, \dots, r_k$ are all odd.

Example 9.7. When $d = 3$, one example of an LS-chain is

$$y_1 e_2 - z_1 e_3, \quad y_2 e_3 - x_2 e_5, \quad y_3 e_5 - x_1 e_7.$$

This corresponds to the unique LS-path between the generators of J corresponding to e_2 and e_7, which are $x_1 x_2 z_1$ and $y_1 y_2 y_3$ respectively. The path goes $x_1 x_2 z_1$, $x_1 x_2 y_1$, $x_1 y_1 y_2$, $y_1 y_2 y_3$, which indeed corresponds to e_2, e_3, e_5, e_7 (see Figure 6).

Notice that the length of an LS-chain is at most d, and that condition (3) along with the structure of the generators implies that $r_{i+1} - r_i = 2$ for all $2 \leq i \leq k-1$.

Lemma 9.8. *Let generators $q_1, \dots, q_k$ be an LS-chain (still with $k \geq 2$), and let $j_{r_1}, \dots, j_{r_k}, j_{s_k}$ be the generators of J corresponding to $e_{r_1}, \dots, e_{r_k}, e_{s_k}$. Then:*

(1) $c_1 \mid j_{s_k}$.

(2) Set $w_1 := \frac{j_{s_k}}{c_1}$. Then $c_2 \mid w_1$. In general, for $1 \leq i \leq k-1$, w_i is divisible by c_{i+1}, where we inductively define $w_{i+1} := \frac{d_i}{c_{i+1}} \cdot w_i$.

(3) $\sum_{i=1}^{k} w_i q_i = 0$.

Proof. For (1), we use the construction of the generators, the definition of the e_r's, the condition $r_1 < r_2 = s_1$, and the fact that r_2 is odd. We see that $q_1 = c_1 e_{r_1} - d_1 e_{r_2}$ corresponds to the linear syzygy between j_{r_1} and j_{r_2}, which means r_1 is either $r_2 - 1$ or $r_2 - 2$. Hence j_{r_2} is a multiple of $y_{\frac{r_2+1}{2}-1}$, and j_{r_1} is not. Also, the existence of the linear syzygy between j_{r_1} and j_{r_2} means that $c_1 = y_{\frac{r_2+1}{2}-1}$. Then because $s_k > r_2$, it follows that j_{s_k} also is a multiple of $y_{\frac{r_2+1}{2}-1}$, hence (1).

For (2), we first consider the case when $1 \leq i \leq k-2$. For each such i, $s_{i+1} = r_{i+2}$ is odd, so the same logic as above applies. In other words, $q_{i+1} = c_{i+1} e_{r_{i+1}} - d_{i+1} e_{r_{i+2}}$ corresponds to the linear syzygy between $j_{r_{i+1}}$ and $j_{r_{i+2}}$, so $j_{r_{i+2}}$ is a multiple of $y_{\frac{r_{i+2}+1}{2}-1}$, and $j_{r_{i+1}}$ is not. This implies that $c_{i+1} = y_{\frac{r_{i+2}+1}{2}-1}$. But recall that we defined $w_1 = \frac{j_{s_k}}{c_1} = \frac{j_{s_k}}{y_{\frac{r_2+1}{2}-1}}$. By the constructions and the fact that $s_k > r_k > \ldots > r_1$, we know that j_{s_k} is divisible by each of the $y_{\frac{r_{i+1}+1}{2}-1} = c_i$ for $1 \leq i \leq k-1$, and all of these terms are distinct as the r_i are distinct odd integers.

It remains to discuss the case $i = k-1$. If s_k is odd, then we can proceed as above. If s_k is even, then j_{s_k} is divisible by $z_{\frac{s_k}{2}}$, and since r_k is odd, j_{r_k} is not. Since $q_k = c_k e_{r_k} - d_k e_{s_k}$ corresponds to the linear syzygy between j_{r_k} and j_{s_k}, we conclude that $r_k = z_{\frac{s_k}{2}}$. It follows that w_{k-1} is divisible by r_k, since w_{k-1} is a multiple of a quotient of j_{s_k}, where the quotient is obtained by only dividing out y-indeterminants (as seen in the previous paragraph). This proves (2).

We turn to (3). Notice that if we expand out the sum in full, we have terms $-w_i d_i e_{r_{i+1}} + w_{i+1} c_{i+1} e_{r_{i+1}}$ for all $1 \leq i \leq k-1$, so these all cancel out by the definition of the w's. Hence

$$\sum_{i=1}^{k} w_i q_i = w_1 c_1 e_{r_1} - w_k d_k e_{s_k}.$$

By definition, $w_1 c_1 = j_{s_k}$. But our LS-chain of linear syzygies implies that $j_{r_i} = j_{r_{i+1}} \cdot \frac{d_i}{c_i}$ for each $1 \leq i \leq k-1$, and $j_{r_k} \cdot \frac{d_k}{c_k} = j_{s_k}$. Hence w_k turns out to be exactly $j_{r_1} d_k$, so $\sum_{i=1}^{k} w_i q_i = j_{s_k} e_{r_1} - j_{r_1} e_{s_k}$. But this is in F_0, so it vanishes. $\square$

With this setup, we are ready to begin the proof of Theorem 9.3. As mentioned before, the idea is to construct the b_i inductively. Hence:

Proposition 9.9. *Let $1 \leq i \leq d$. Suppose we are given initial data $b'_1, \ldots, b'_{2i-1} \in S/J$ such that:*

(1) If $i \geq 2$, then for $1 \leq k \leq i-1$,

$$z_k b'_{2k-1} - x_{d-k+1} b'_{2k} = a_{3k-2}, \quad y_k b'_{2k-1} - x_{d-k+1} b'_{2k+1} = a_{3k-1}, \quad y_k b'_{2k} - z_k b'_{2k+1} = a_{3k}.$$

(2) $a_{3i-2} - z_i b'_{2i-1}$ is a multiple of x_{d-i+1}.

Then we may find $b_1, \ldots, b_{2i+1} \in S/J$ such that:

I. For $1 \leq k \leq i$,

$$z_k b_{2k-1} - x_{d-k+1} b_{2k} = a_{3k-2}, \quad y_k b_{2k-1} - x_{d-k+1} b_{2k+1} = a_{3k-1}, \quad y_k b_{2k} - z_k b_{2k+1} = a_{3k}.$$

II. If $i \leq d-1$, then $a_{3i+1} - z_{i+1} b_{2i+1}$ is a multiple of x_{d-i}.

Proof. Part I.

Recall from Lemma 9.5 that we may write

$$a_{3i-2} = x_{d-i+1} f_{3i-2} + z_i h_{3i-2}, \quad a_{3i-1} = x_{d-i+1} f_{3i-1} + y_i g_{3i-1}, \quad a_{3i} = y_i g_{3i} + z_i h_{3i},$$

for some $f_k, g_k, h_k \in S/I$, and that

$$-y_i(x_{d-i+1} f_{3i-2} + z_i h_{3i-2}) + z_i(x_{d-i+1} f_{3i-1} + y_i g_{3i-1}) = x_{d-i+1}(y_i g_{3i} + z_i h_{3i}). \tag{1}$$

Since we are given that $a_{3i-2} - z_i b'_{2i-1}$ is a multiple of x_{d-i+1}, we may assume that $h_{3i-2} = b'_{2i-1}$ above. Temporarily set $b'_{2i} := -f_{3i-2}$, so $z_i b'_{2i-1} - x_{d-i+1} b'_{2i} = a_{3i-2}$.

We next claim that $a_{3i-1} - y_i b'_{2i-1}$ is a multiple of x_{d-i+1}. From Equation 1, we have

$$z_i(a_{3i-1} - y_i b'_{2i-1}) = x_{d-i+1}(a_{3i} + y_i f_{3i-2}),$$

and we know that $a_{3i-1} - y_i b'_{2i-1}$ is in (x_{d-i+1}, y_i). Taking lifts in S, we also see that (a lift of) $a_{3i-1} - y_i b'_{2i-1}$ is in $((x_{d-i+1}) + J) : (z_i)$. Proposition 1.2.2 in [HH11] tells us that $((x_{d-i+1}) + J) : (z_i)$ is generated by x_{d-i+1}, J, and $x_1 \ldots x_{d-i} y_1 \ldots y_{i-1}$, since the only minimal generator of $(x_{d-i+1}) + J$ divisible by z_i is $x_1 \ldots x_{d-i} y_1 \ldots y_{i-1} z_i$. Therefore $a_{3i-1} - y_i b'_{2i-1}$ is in both $(x_{d-i+1}) + (y_i) + J$ and $(x_{d-i+1}) + (x_1 \ldots x_{d-i} y_1 \ldots y_{i-1}) + J$, and Proposition 1.2.1 in [HH11] tells us that the intersection of those two monomial ideals is exactly $(x_{d-i+1}) + J + (\mathrm{lcm}(y_i, x_1 \ldots x_{d-i} y_1 \ldots y_{i-1})) = (x_{d-i+1}) + J$, as $x_1 \ldots x_{d-i} y_1 \ldots y_i$ is already in J. In other words, $a_{3i-1} - y_i b'_{2i-1}$ is indeed a multiple of x_{d-i+1} in S/J.

So let b'_{2i+1} be an element of S/J such $-x_{d-i+1}b'_{2i+1} = a_{3i-1} - y_i b'_{2i-1}$. Then we indeed have $y_i b'_{2i-1} - x_{d-i+1}b'_{2i+1} = a_{3i-1}$. Now, we know from Equation 1 (and the fact that $h_{3i-2} = b'_{2i-1}$) that

$$x_{d-i+1}(y_i b'_{2i} - z_i b'_{2i+1}) = -x_{d-i+1} y_i f_{3i-2} + z_i(a_{3i-1} - y_i b'_{2i-1}) = x_{d-i+1} a_{3i}.$$

In other words, the difference $a_{3i} - (y_i b'_{2i} - z_i b'_{2i+1})$ is an element of S/J killed by x_{d-i+1}, and is also in (y_i, z_i). Taking a lift $t \in S$, we see that t is in both the monomial ideals $J : (x_{d-i+1})$ and $J + (y_i, z_i)$. Hence t is in an ideal $J + J_1$, where J_1 is an ideal generated by monomials $\ell \notin J$ such that either y_i or z_i divides ℓ, and $x_{d-i+1}\ell \in J$. Moreover, in light of item (2) of Lemma 9.4, it follows that if y_i (resp. z_i) divides ℓ, then $x_{d-i+1}\left(\frac{\ell}{y_i}\right)$ (resp. $x_{d-i+1}\left(\frac{\ell}{z_i}\right)$) also lands in J. Indeed, if $x_{d-i+1}\ell \in J$, then it is divisible by some monomial $\mathbf{m} \in J$; say $\mathbf{m}\mathbf{m}' = x_{d-i+1}\ell$. If $y_i \nmid \mathbf{m}$, then $y_i \mid \mathbf{m}'$, upon which $\mathbf{m}$ still divides $x_{d-i+1}\left(\frac{\ell}{y_i}\right)$. If $y_i \mid \mathbf{m}$, then $x_{d-i+1} \nmid \mathbf{m}$, so that $\mathbf{m} \mid \ell$, contradicting $\ell \notin J$.

Now, passing back to the quotient, we see that $a_{3i} - (y_i b'_{2i} - z_i b'_{2i+1})$ is an S/J-linear combination of the aforementioned ℓ's. In particular, we may combine terms and write $a_{3i} - (y_i b'_{2i} - z_i b'_{2i+1}) = y_i r - z_i s$, where both $r, s \in S/J$ are killed by x_{d-i+1}. Therefore replace b'_{2i} with $b'_{2i} + r$ and b'_{2i+1} with $b'_{2i+1} + s$.

With these new values of b'_{2i} and b'_{2i+1}, we still have $z_i b'_{2i-1} - x_{d-i+1} b'_{2i} = a_{3i-2}$ and $y_i b'_{2i-1} - x_{d-i+1} b'_{2i+1} = a_{3i-1}$, since the adjusted value of b'_{2i} (resp. b'_{2i+1}) differs from the old value by some element killed by x_{d-i+1}. Moreover, $y_i b'_{2i} - z_i b'_{2i+1} = a_{3i}$ by construction. Therefore we have $b'_1, \ldots, b'_{2i+1}$ satisfying item I.

Part II.

Now, suppose $i < d$. We will need to adjust all of the $b'_1, \ldots, b'_{2i+1}$ in a way that satisfies item II, but also preserves the equalities in item I.

We want to consider the following LS-chain:

$$y_1 e_1 - x_d e_3, \quad y_2 e_3 - x_{d-1} e_5, \quad \ldots, \quad y_i e_{2i-1} - x_{d-i+1} e_{2i+1}, \quad z_{i+1} e_{2i+1} - x_{d-i} e_{2i+2}.$$

Note that applying φ to these generators gives $a_2, a_5, \ldots, a_{3i-1}, a_{3i+1}$.

We now apply Lemma 9.8. This gives us a vanishing linear combination:

$$\begin{aligned}&(x_1 \ldots x_{d-i-1} y_2 \ldots y_i z_{i+1})(y_1 e_1 - x_d e_3) + (x_1 \ldots x_{d-i-1} x_d y_3 \ldots y_i z_{i+1})(y_2 e_3 - x_{d-1} e_5) + \ldots \\ &+ (x_1 \ldots x_{d-i-1} x_{d-i+2} \ldots x_d z_{i+1})(y_i e_{2i-1} - x_{d-i+1} e_{2i+1}) + (x_1 \ldots x_{d-i-1} x_{d-i+1} \ldots x_d)(z_{i+1} e_{2i+1} - x_{d-i} e_{2i+2}) = 0.\end{aligned} \tag{2}$$

Upon applying φ, we get

$$\begin{aligned}&(x_1 \ldots x_{d-i-1} y_2 \ldots y_i z_{i+1}) a_2 + (x_1 \ldots x_{d-i-1} x_d y_3 \ldots y_i z_{i+1}) a_5 + \ldots \\ &+ (x_1 \ldots x_{d-i-1} x_{d-i+2} \ldots x_d z_{i+1}) a_{3i-1} + (x_1 \ldots x_{d-i-1} x_{d-i+1} \ldots x_d) a_{3i+1} = 0.\end{aligned} \tag{3}$$

For $a_2, a_5, \ldots, a_{3i-1}$, we may expand each in terms of b'_is (i.e. $a_{3k-1} = y_k b'_{2k-1} - x_{d-k+1} b'_{2k+1}$ for $1 \le k \le i$), and for a_{3i+1}, we may write it as $x_{d-i} f_{3i+1} + z_{i+1} h_{3i+1}$, due to Lemma 9.5. After mass cancellations, Equation 3 becomes

$$\begin{aligned}&(x_1 \ldots x_{d-i-1} y_2 \ldots y_i z_{i+1}) y_1 b'_1 + (x_1 \ldots x_{d-i-1} x_{d-i+2} \ldots x_d z_{i+1})(-x_{d-i+1} b'_{2i+1}) \\ &+ (x_1 \ldots x_{d-i-1} x_{d-i+1} \ldots x_d)(x_{d-i} f_{3i+1} + z_{i+1} h_{3i+1}) = 0.\end{aligned}$$

We recognize that $x_1 \ldots x_{d-i-1} y_1 \ldots y_i z_{i+1}$ and $x_1 \ldots x_d$ both vanish in the quotient ring S/J, so the above simplifies to

$$(x_1 \ldots x_{d-i-1} x_{d-i+1} \ldots x_d z_{i+1})(h_{3i+1} - b'_{2i+1}) = 0. \tag{4}$$

As before, consider a lift of $h_{3i+1} - b'_{2i+1}$ inside S, which must be in the colon ideal

$$J : (x_1 \ldots x_{d-i-1} x_{d-i+1} \ldots x_d z_{i+1}).$$

Recall from Lemma 9.4 that every generator of J is either divisible by x_{d-i} or y_i, and by the construction of J, if it is divisible by y_i, then it is divisible by $y_1 \dots y_i$. Then because $x_1 \dots x_{d-i-1} x_{d-i+1} \dots x_d z_{i+1}$ is coprime to both x_{d-i} and $y_1 \dots y_i$, Proposition 1.2.2 in [HH11] shows that this colon ideal is contained in $(x_{d-i}, y_1 \dots y_i) \supseteq J$. Hence $h_{3i+1} - b'_{2i+1}$ equals $x_{d-i}t + y_1 \dots y_i u$ for some $t, u \in S/J$.

Define now $b_{2i+1} := b'_{2i+1} + y_1 \dots y_i u$. Then

$$a_{3i+1} - z_{i+1} b_{2i+1} = (x_{d-i} f_{3i+1} + z_{i+1} h_{3i+1}) - z_{i+1}(b'_{2i+1} + y_1 \dots y_i u) = x_{d-i} f_{3i+1} + z_{i+1}(x_{d-i} t),$$

so $a_{3i+1} - z_{i+1} b_{2i+1}$ is a multiple of x_{d-i}. Recalling that we have equations

$$z_i b'_{2i-1} - x_{d-i+1} b'_{2i} = a_{3i-2}, \quad y_i b'_{2i-1} - x_{d-i+1} b'_{2i+1} = a_{3i-1}, \quad y_i b'_{2i} - z_i b'_{2i+1} = a_{3i},$$

we may set $b_{2i-1} := b'_{2i-1} + y_1 \dots y_{i-1} x_{d-i+1} u$ and $b_{2i} := b'_{2i} + y_1 \dots y_{i-1} z_i u$ to obtain

$$z_i b_{2i-1} - x_{d-i+1} b_{2i} = a_{3i-2}, \quad y_i b_{2i-1} - x_{d-i+1} b_{2i+1} = a_{3i-1}, \quad y_i b_{2i} - z_i b_{2i+1} = a_{3i}.$$

Next, we have equations

$$z_{i-1} b'_{2i-3} - x_{d-i+2} b'_{2i-2} = a_{3i-5}, \quad y_{i-1} b'_{2i-3} - x_{d-i+2} b'_{2i-1} = a_{3i-4}, \quad y_{i-1} b'_{2i-2} - z_{i-1} b'_{2i-1} = a_{3i-3},$$

so we may set $b_{2i-3} := b'_{2i-3} + y_1 \dots y_{i-2} x_{d-i+1} x_{n-i+2} u$ and $b_{2i-2} := b'_{2i-2} + y_1 \dots y_{i-2} x_{d-i+1} z_{i-1} u$ to obtain

$$z_{i-1} b_{2i-3} - x_{d-i+2} b_{2i-2} = a_{3i-5}, \quad y_{i-1} b_{2i-3} - x_{d-i+2} b_{2i-1} = a_{3i-4}, \quad y_{i-1} b_{2i-2} - z_{i-1} b_{2i-1} = a_{3i-3}.$$

This procedure is easily repeated inductively: in general for $0 \le k \le i$, we set

$$b_{2k+1} = b'_{2k+1} + y_1 \dots y_k x_{d-i+1} \dots x_{d-k} u, \qquad b_{2k} = b'_{2k} + y_1 \dots y_{k-1} x_{d-i+1} \dots x_{d-k} z_k u.$$

Then the equations

$$z_k b'_{2k-1} - x_{d-k+1} b'_{2k} = a_{3k-2}, \quad y_k b'_{2k-1} - x_{d-k+1} b'_{2k+1} = a_{3k-1}, \quad y_k b'_{2k} - z_k b'_{2k+1} = a_{3k}$$

are easily seen to be preserved; i.e. we have

$$z_k b_{2k-1} - x_{d-k+1} b_{2k} = a_{3k-2}, \quad y_k b_{2k-1} - x_{d-k+1} b_{2k+1} = a_{3k-1}, \quad y_k b_{2k} - z_k b_{2k+1} = a_{3k}.$$

So our construction of the b_k's satisfies both item I and item II. □

Proof of Theorem 9.3. By Lemma 9.5, we may write $a_1 = x_d f_1 + z_1 h_1$ and set $b'_1 := h_1$, so $a_1 - z_1 b'_1$ is a multiple of x_d in S/J. By applying Proposition 9.9 inductively, we may find $b_1, \dots, b_{2d+1} \in S/J$ such that for all $1 \le i \le d$,

$$z_i b_{2i-1} - x_{d-i+1} b_{2i} = a_{3i-2} = \varphi(z_i e_{2i-1} - x_{d-i+1} e_{2i}), \quad y_i b_{2i-1} - x_{d-i+1} b_{2i+1} = a_{3i-1} = \varphi(y_i e_{2i-1} - x_{d-i+1} e_{2i+1}),$$
$$y_i b_{2i} - z_i b_{2i+1} = a_{3i} = \varphi(y_i e_{2i} - z_i e_{2i+1}).$$

Hence the map $\psi : (S/J)^{2d+1} \to S/J$ sending e_i to b_i induces φ, and the map $\mathrm{Hom}_{S/J}(F/JF, S/J) \to \mathrm{Hom}_{S/J}(Q/F_0, S/J)$ is surjective as desired. □

Note that this result is quite particular to this specific polarization of $I_{3,n}$. For instance, one can define another polarization of $I_{3,3}$ that is similar to $J_{3,3}$:

$$I' := (x_1x_2x_3, x_1x_2y_1, x_1x_2z_1, x_1y_1y_2, x_1y_1z_2, y_1y_2y_3, y_1y_2z_1).$$

The graph of linear syzygies of I' is the same as that of $J_{3,3}$; the only difference is that I' has the generator $y_1y_2z_1$ instead of $y_1y_2z_3$. But $T^2_{I'} \neq 0$; a computation with Macaulay2 shows that it has dimension 3.

In fact, the same computation can be generalized to $J_{n,d}$ with $n \ge 3$, but the details will be messier (but in the same spirit), so we will omit them. In other words,

Theorem 9.10. *For $n \geq 3$, $T^2_{J_{n,d}} = 0$. In particular, $J_{n,d}$ determines a smooth point $x_{n,d}$ on its associated Hilbert scheme.*

Since we now know that $J_{n,d}$ corresponds to a smooth point $x_{n,d}$ on the associated Hilbert scheme, it is natural to ask for the dimension of the tangent space at $x_{n,d}$. We provide this computation below.

Proposition 9.11. *For $n \geq 3$ and $d \geq 2$, the dimension of the tangent space at $x_{n,d}$ is $\dim_{k(x_{n,d})} T_{x_{n,d}} = d(d-1)(n^2+n-1)$.*

Because $S/J := S/J_{n,d}$ visibly has depth at least 2, the desired dimension is exactly $\dim_k \operatorname{Hom}_S(J, S/J)_0$, the dimension of the degree-preserving S-module maps $J \to S/J$ ([Loh13], Proposition 2.4). We will be investigating this latter dimension.

Let $\varphi \in \operatorname{Hom}_S(J, S/J)_0$. We discuss conditions that φ must satisfy. Notice first that such maps φ are exactly characterized by the criterion that the (homogeneous degree-d) images of φ on the generators of J satisfy the corresponding linear syzygy relations. Indeed, if $0 \to Q \to F \to J \to 0$ is a free presentation of J, with Q the submodule of F generated by the linear syzygies of J, then $\operatorname{Hom}_S(J, S/J)_0$ is exactly the kernel of $\operatorname{Hom}_S(F, S/J)_0 \to \operatorname{Hom}_S(Q, S/J)_0$.

To begin, we prove a few useful lemmas.

Lemma 9.12. *Suppose that $2 \leq i \leq d$ is such that $\varphi(x_{1,1} \dots x_{1,d})$ is annihilated by $x_{2,1} \dots x_{2,i}$, but not by $x_{2,1} \dots x_{2,i-1}$. Pick a homogeneous representative of $\varphi(x_{1,1} \dots x_{1,d})$, and let $\mathbf{m}_1, \dots, \mathbf{m}_k \in S$ be the monomials in that representative that are not in $J : (x_{2,1} \dots x_{2,i-1})$. Then the $\mathbf{m}_r$ are contained in $(x_{1,d-i+2} \dots x_{1,d})$.*

Note that $x_{2,1} \dots x_{2,d} = 0$, so such an i surely exists.

Proof. We have a linear syzygy

$$x_{2,1} \dots x_{2,i-1} \varphi(x_{1,1} \dots x_{1,d}) = x_{1,d-i+2} \dots x_{1,d} \varphi(x_{1,1} \dots x_{1,d-i+1} x_{2,1} \dots x_{2,i-1}).$$

Lift this equation up to S, so we may write

$$x_{2,1} \dots x_{2,i-1}(\mathbf{m}_1 + \dots + \mathbf{m}_k) + j = x_{1,d-i+2} \dots x_{1,d} \varphi(x_{1,1} \dots x_{1,d-i+1} x_{2,1} \dots x_{2,i-1}), \tag{5}$$

where $j \in J$. Note that because we assume $\varphi(x_{1,1} \dots x_{1,d})$ is not annihilated by $x_{2,1} \dots x_{2,i-1}$, there is at least one such $\mathbf{m}_r$. Since none of the monomial terms on the left hand side of Equation 5 cancel after expanding, it follows that each of the $x_{2,1} \dots x_{2,i-1} \mathbf{m}_r$ terms are divisible by $x_{1,d-i+2} \dots x_{1,d}$ in S, which implies the statement. ☐

Lemma 9.13. *Consider the same assumptions as in Lemma 9.12, but now allowing $i = 1$ (in this case, $\varphi(x_{1,1} \dots x_{1,d})$ is annihilated by $x_{2,1}$). Suppose furthermore that the $\mathbf{m}_r$ are not in $J : (x_{2,1} \dots x_{2,i-1} x_{3,i})$ either. Then the $\mathbf{m}_r$ are contained in $(x_{1,d-i+1} x_{1,d-i+2} \dots x_{1,d})$.*

Proof. We have a linear syzygy

$$x_{2,1} \dots x_{2,i-1} x_{3,i} \varphi(x_{1,1} \dots x_{1,d}) = x_{1,d-i+1} \dots x_{1,d} \varphi(x_{1,1} \dots x_{1,d-i} x_{2,1} \dots x_{2,i-1} x_{3,i}).$$

Then conclude as in Lemma 9.12. ☐

Remark 9.14. Similarly, we can show that if $2 \le i \le d$ is such that $\varphi(x_{2,1} \dots x_{2,d})$ is annihilated by $x_{1,1} \dots x_{1,i}$ but not $x_{1,1} \dots x_{1,i-1}$, then for each monomial $\mathbf{m}_r \notin J : (x_{1,1} \dots x_{1,i-1})$ that is in a homogeneous representative of $\varphi(x_{2,1} \dots x_{2,d})$, we have $\mathbf{m}_r \in (x_{2,d-i+2} \dots x_{2,d})$. If $\mathbf{m}_r \notin J : (x_{1,1} \dots x_{1,i-1} x_{3,d-i+1})$ (and allowing $i = 1$), then we may even say $\mathbf{m}_r \in (x_{2,d-i+1} \dots x_{2,d})$.

Our next key idea is that the images of $x_{1,1} \dots x_{1,d}$ and $x_{2,1} \dots x_{2,d}$ come very close to determining the entire map $\varphi \in \mathrm{Hom}_S(J, S/J)_0$, up to the addition of possible annihilated elements.

Lemma 9.15. *Suppose $\varphi \in Hom_S(J, S/J)_0$ is such that φ vanishes at both $x_{1,1} \dots x_{1,d}$ and $x_{2,1} \dots x_{2,d}$. Then $\varphi(x_{1,1} \dots x_{1,d-i} x_{2,1} \dots x_{2,i}) = 0$ for all $0 \le i \le d$.*

Proof. Because we have the linear syzygies

$$x_{2,1} \dots x_{2,i} \varphi(x_{1,1} \dots x_{1,d}) = x_{1,d-i+1} \dots x_{1,d} \varphi(x_{1,1} \dots x_{1,d-i} x_{2,1} \dots x_{2,i})$$
$$x_{1,1} \dots x_{1,d-i} \varphi(x_{2,1} \dots x_{2,d}) = x_{2,i+1} \dots x_{2,d} \varphi(x_{1,1} \dots x_{1,d-i} x_{2,1} \dots x_{2,i}),$$

it follows that $\varphi(x_{1,1} \dots x_{1,d-i} x_{2,1} \dots x_{2,i})$ is killed by both $x_{1,d-i+1} \dots x_{1,d}$ and $x_{2,i+1} \dots x_{2,d}$. The first condition implies that $\varphi(x_{1,1} \dots x_{1,d-i} x_{2,1} \dots x_{2,i})$ is a multiple of $x_{1,1} \dots x_{1,d-i}$ in S/J; the second implies that it is a multiple of $x_{2,1} \dots x_{2,i}$. Hence $\varphi(x_{1,1} \dots x_{1,d-i} x_{2,1} \dots x_{2,i})$ vanishes. $\square$

We are now in a position to describe the possible maps $\varphi \in \mathrm{Hom}_S(J, S/J)_0$. The main idea is as follows. We know that if it is nonzero, $\varphi(x_{1,1} \dots x_{1,d})$ has a representative that is a sum of degree-d monomials, each of which are killed by some $x_{2,1} \dots x_{2,i}$. Using Lemmas 9.12 and 9.13, we will seek to describe all such monomials, and show that for each, there is a "basic" map in $\mathrm{Hom}_S(J, S/J)_0$ sending $x_{1,1} \dots x_{1,d}$ to that monomial, and $x_{2,1} \dots x_{2,d}$ to 0. We will do something analogous for $\varphi(x_{2,1} \dots x_{2,d})$. The upshot is that by Lemma 9.15, we know that any map in $\mathrm{Hom}_S(J, S/J)_0$ is equal to a k-linear combination of such basic maps, except for possibly differing at the $x_{1,1} \dots x_{1,d-i} x_{2,1} \dots x_{2,i-1} x_{k,i}$ for $3 \le k \le n$. However, the maps in $\mathrm{Hom}_S(J, S/J)_0$ that vanish at the $x_{1,1} \dots x_{1,d-i} x_{2,1} \dots x_{2,i}$ will turn out to be easy to describe.

First, for a fixed value $2 \le i \le d$, consider the nonzero degree-d monomials $\mathbf{m}$ (in S/J) killed by $x_{2,1} \dots x_{2,i}$, not killed by $x_{2,1} \dots x_{2,i-1}$, and in $(x_{1,d-i+2} \dots x_{1,d})$. We split into two cases:

Case I: $x_{2,1} \dots x_{2,i-1} x_{3,i}$ kills $\mathbf{m}$.

Up to scalars, we claim that such monomials look like

$$x_{1,1} \dots x_{1,d-i} x_{1,d-i+2} \dots x_{1,d} a,$$

where a is some indeterminant that we will impose restrictions on later. Let us consider the situation inside S. We want to find the degree-d monomials $\mathbf{m}$, divisible by $x_{1,d-i+2} \dots x_{1,d}$, in $J : (x_{2,1} \dots x_{2,i})$ and $J : (x_{2,1} \dots x_{2,i-1} x_{3,i})$ but not in $J : (x_{2,1} \dots x_{2,i-1})$. We know that $J : (x_{2,1} \dots x_{2,i})$ is generated by terms of the form $\frac{j}{\gcd(j, x_{2,1} \dots x_{2,i})}$, and due to the structure of J, if $x_{2,i} \nmid \gcd(j, x_{2,1} \dots x_{2,i})$, then $x_{2,1} \dots x_{2,i-1} \cdot \frac{j}{\gcd(j, x_{2,1} \dots x_{2,i})} \in J$. Therefore $\mathbf{m}$ is a multiple of something that looks like $j' := \frac{j}{x_{2,1} \dots x_{2,i}}$, where j is a generator of J divisible by $x_{2,i}$ (and hence $x_{2,1} \dots x_{2,i}$). Therefore j is equal to $x_{1,1} \dots x_{1,d-k-1} x_{2,1} \dots x_{2,k} b$, where $i \le k \le d$ and b is one of $x_{1,d-k}, x_{3,k+1}, \dots, x_{n,k+1}$ when $k \le d-1$ (when $k = d$, $b = 1$). Hence j' looks like

$$x_{1,1} \dots x_{1,d-k-1} x_{2,i+1} \dots x_{2,k} b,$$

and $\mathbf{m}$ is divisible by

$$x_{1,1} \dots x_{1,d-k-1} x_{1,d-i+2} \dots x_{1,d} x_{2,i+1} \dots x_{2,k} b.$$

This is $d-1$ indeterminants, so up to scalars, $\mathbf{m}$ is the above monomial times some indeterminant a. But if $x_{2,1}\dots x_{2,i-1}x_{3,i}$ kills $\mathbf{m}$ inside S/J, then

$$x_{1,1}\dots x_{1,d-k-1}x_{1,d-i+2}\dots x_{1,d}x_{2,i}\dots x_{2,i-1}x_{2,i+1}\dots x_{2,k}x_{3,i}ba \in J.$$

The only way this is possible is if the above is a multiple of $x_{1,1}\dots x_{1,d-i}$. In particular, $k = i$, $b = x_{d-k} = x_{d-i}$ (when $i = d$, this means $b = 1$), and a can be any indeterminant such that $x_{1,1}\dots x_{1,d-i}x_{1,d-i+2}\dots x_{1,d}a \notin J$ (there is another case with $k = i+1, b = x_{d-k} = x_{d-i-1}$, and $a = x_{d-i}$, but this is subsumed). This gives the original claim, and a can be any of the $nd-1$ indeterminants in S besides $x_{1,d-i+1}$.

We now build a map $\psi \in \mathrm{Hom}_S(J, S/J)_0$ with $\psi(x_{1,1}\dots x_{1,d}) = \mathbf{m} = x_{1,1}\dots x_{1,d-i}x_{1,d-i+2}\dots x_{1,d}a$. This is not so hard:

- For $0 \le k \le i-1$, ψ sends $x_{1,1}\dots x_{1,d-k}x_{2,1}\dots x_{2,k}$ to $x_{1,1}\dots x_{1,d-i}x_{1,d-i+2}\dots x_{1,d-k}x_{2,1}\dots x_{2,k}a$.
- For $0 \le k \le i-1$ and $3 \le r \le n$, ψ sends $x_{1,1}\dots x_{1,d-k}x_{2,1}\dots x_{2,k-1}x_{r,k}$ to $x_{1,1}\dots x_{1,d-i}x_{1,d-i+2}\dots x_{1,d-k}x_{2,1}\dots x_{2,k-1}x_{r,k}a$.
- ψ is 0 on all other generators of J. In particular, ψ is 0 on $x_{2,1}\dots x_{2,d}$.

For each value $2 \le i \le d$, there are $nd-1$ such maps, and they are mutually k-linearly independent due to the $\mathbf{m}$ visibly being linearly independent (and $\psi(x_{1,1}\dots x_{1,d}) = \mathbf{m}$).

Case II: $x_{2,1}\dots x_{2,i-1}x_{3,i}$ does not kill $\mathbf{m}$, and $\mathbf{m} \in (x_{1,d-i+1}\dots x_{1,d})$.

Up to scalars, we claim that such monomials look like

$$x_{1,1}\dots x_{1,d-k-1}x_{1,d-i+1}\dots x_{1,d}x_{2,i+1}\dots x_{2,k}b,$$

for $i \le k \le d$ and b some indeterminant that we will impose restrictions on later. As above, $\mathbf{m}$ is a multiple of something that looks like $j' := \frac{j}{x_{2,1}\dots x_{2,i}}$, where j is a generator of J divisible by $x_{2,i}$ (and hence $x_{2,1}\dots x_{2,i}$). Therefore $j = x_{1,1}\dots x_{1,d-k-1}x_{2,1}\dots x_{2,k}b$, where $i \le k \le d$ and b is one of $x_{1,d-k}, x_{3,k+1}, \dots, x_{n,k+1}$ when $k \le d-1$ (when $k = d$, $b = 1$). Hence j' looks like

$$x_{1,1}\dots x_{1,d-k-1}x_{2,i+1}\dots x_{2,k}b,$$

and $\mathbf{m}$ is divisible by

$$x_{1,1}\dots x_{1,d-k-1}x_{1,d-i+1}\dots x_{1,d}x_{2,i+1}\dots x_{2,k}b.$$

This is d indeterminants, so up to scalars, $\mathbf{m}$ is exactly this type of monomial, which is the claim. Let's see how many such $\mathbf{m}$ there are. For each fixed i, there are $d-i$ choices of $i \le k \le d-1$ and $n-1$ choices of b for such k. When $k = n$ and $b = 1$, there is exactly one choice of b. However, note that the choice $k = i$ brings us back to Case I (here $\mathbf{m} = x_{1,1}\dots x_{1,d-i-1}x_{1,d-i+1}\dots x_{1,d}b$, which is subsumed under Case I), and so does the choice $k = i+1$, $b = x_{1,d-k} = x_{1,d-i-1}$ (here $\mathbf{m} = x_{1,1}\dots x_{1,d-i-1}x_{1,d-i+1}\dots x_{1,d}x_{2,i+1}$, again subsumed under Case I). So we've only found $(n-1)(d-i)+1-(n-1)-1 = (n-1)(d-i-1)$ new possible values of $\mathbf{m}$.

Again, we build a map $\psi \in \mathrm{Hom}_S(J, S/J)_0$ with

$$\psi(x_{1,1}\dots x_{1,d}) = \mathbf{m} = x_{1,1}\dots x_{1,d-k-1}x_{1,d-i+1}\dots x_{1,d}x_{2,i+1}\dots x_{2,k}b$$

as follows:

- For $0 \le l \le i-1$,

  $$\psi(x_{1,1}\dots x_{1,d-l}x_{2,1}\dots x_{2,l}) = x_{1,1}\dots x_{1,d-k-1}x_{1,d-i+1}\dots x_{1,d-l}x_{2,1}\dots x_{2,l}x_{2,i+1}\dots x_{2,k}b.$$

- For $0 \le l \le i$ and $3 \le r \le n$, ψ sends $x_{1,1}\dots x_{1,d-l}x_{2,1}\dots x_{2,l-1}x_{r,l}$ to $x_{1,1}\dots x_{1,d-k-1}x_{1,d-i+1}\dots x_{1,d-l}x_{2,1}\dots x_{2,l-1}x_{2,i+1}\dots x_{2,k}x_{r,l}b$.

- ψ is 0 on all other generators of J. In particular, ψ is 0 on $x_{2,1}\dots x_{2,d}$.

For each value $2 \le i \le d$, there are $(n-1)(d-i)$ new maps, and they are visibly mutually k-linearly independent.

Using an analogous procedure, we may describe all degree-d monomials $\mathbf{m}$ satisfying the hypotheses in Remark 9.14, and for each $\mathbf{m}$, we may build a map $\psi \in \mathrm{Hom}_S(J, S/J)_0$ with $\psi(x_{2,1}\dots x_{2,d}) = \mathbf{m}$ and $\psi(x_{1,1}\dots x_{1,d}) = 0$. We get the same count for the number of such k-linearly independent maps. So far, we have found a total of

$$2\Big((d-1)(nd-1)+(n-1)(d-3)+(n-1)(d-4)+\dots+(n-1)\cdot 1\Big) = 2\left((d-1)(nd-1)+(n-1)\frac{(d-3)(d-2)}{2}\right) \tag{6}$$

k-linearly independent maps.

So we have now reduced to the case where we want to describe maps $\varphi \in \mathrm{Hom}_S(J, S/J)_0$ where $\varphi(x_{1,1}\dots x_{1,d})$ is killed by $x_{2,1}$, and $\varphi(x_{2,1}\dots x_{2,d})$ is killed by $x_{1,1}$. First, let $\mathbf{m} \notin J$ be a degree-d monomial in a homogeneous representative of $\varphi(x_{1,1}\dots x_{1,d})$, so φ is also killed by $x_{2,1}$. If $\mathbf{m}$ is killed by $x_{3,1}$, then $\mathbf{m}$ must be a multiple of $x_{1,1}\dots x_{1,d-1}$, so is a scalar multiple of a monomial $x_{1,1}\dots x_{1,d-1}a$, where a is some indeterminant not equal to $x_{1,d}, x_{2,1}, \dots, x_{n,1}$ (as then $\mathbf{m}$ would be in J). As in Case I above, for each of those $nd-n$ possibilities for $\mathbf{m}$, there is $\psi \in \mathrm{Hom}_S(J, S/J)_0$ with $\psi(x_{1,1}\dots x_{1,d}) = \mathbf{m}$ and $\psi(x_{2,1}\dots x_{2,d}) = 0$. On the other hand, if $\mathbf{m}$ is not killed by $x_{3,1}$, then Lemma 9.13 applies (with $i = 1$), and we can conclude as in Case II above: up to a scalar, $\mathbf{m}$ looks like

$$x_{1,1}\dots x_{1,d-k-1}x_{1,d}x_{2,2}\dots x_{2,k}b$$

for $1 \le k \le d$, and b one of $x_{1,d-k}, x_{3,k+1}, \dots, x_{m,k+1}$, unless $k = d$ (in which case $b = 1$). As before, this procedure generates exactly $(n-1)(d-2)$ new monomials $\mathbf{m}$, and for each, we may construct a corresponding map ψ satisfying the usual conditions. All of these $(nd-n)+(n-1)(d-2)$ maps are linearly independent as they take on k-linearly independent values at $x_{1,1}\dots x_{1,d}$.

Again, this procedure can be repeated for monomials in a representative of $\varphi(x_{2,1}\dots x_{2,d})$ that are killed by $x_{1,1}$. Doing this again creates $(nd-n)+(n-1)(d-2)$ new linearly independent maps in $\mathrm{Hom}_S(J, S/J)_0$, and using Equation 6, we now have a total of

$$2\left((nd-n)+(d-1)(nd-1)+(n-1)\frac{(d-2)(d-1)}{2}\right) \tag{7}$$

linearly independent maps.

We have now described all possible monomials in the representatives of $\varphi(x_{1,1}\dots x_{1,d})$ and $\varphi(x_{2,1}\dots x_{2,d})$, where φ is an arbitrary element in $\mathrm{Hom}_S(J, S/J)_0$, and for each such monomial, we have constructed some map ψ that takes on that value at the corresponding generator of J ($x_{1,1}\dots x_{1,d}$ or $x_{2,1}\dots x_{2,d}$). Subtracting off all those maps from φ, we conclude from Lemma 9.15 that we are now in the case where φ vanishes at all $x_{1,1}\dots x_{1,d-i}x_{2,1}\dots x_{2,i}$. We now consider possible values for $\varphi(x_{1,1}\dots x_{1,d-i}x_{2,1}\dots x_{2,i-1}x_{r,i})$, where $3 \le r \le n$. For each, we conclude from the description of the linear syzygies that

$$0 = x_{1,d-i+1}\varphi(x_{1,1}\dots x_{1,d-i}x_{2,1}\dots x_{2,i-1}x_{r,i}) = x_{2,i}\varphi(x_{1,1}\dots x_{1,d-i}x_{2,1}\dots x_{2,i-1}x_{r,i}).$$

Taking a lift $s \in S$ of $\varphi(x_{1,1}\dots x_{1,d-i}x_{2,1}\dots x_{2,i-1}x_{r,i})$, we have

$$s \in (J : (x_{1,d-i+1})) \cap (J : (x_{1,i-1})) \subseteq (J + (x_{1,1}\dots x_{1,d-i})) \cap (J + (x_{2,1}\dots x_{2,i-1})) = J + (x_{1,1}\dots x_{1,d-i}x_{2,1}\dots x_{2,i-1}).$$

In other words, $\varphi(x_{1,1}\dots x_{1,d-i}x_{2,1}\dots x_{2,i-1}x_{r,i})$ is a multiple of $x_{1,1}\dots x_{1,d-i}x_{2,1}\dots x_{2,i-1}$ inside S/J. Hence if $\mathbf{m} \notin J$ is some degree-d monomial in a homogeneous representative, then

$$\mathbf{m} = x_{1,1}\dots x_{1,d-i}x_{2,1}\dots x_{2,i-1}a$$

for some indeterminant a (as always, up to a scalar multiple), which can be any of the nd variables besides $x_{1,d-i+1}, x_{2,i}, x_{3,i} \dots, x_{n,i}$. For each such $\mathbf{m}$, there is a map $\psi \in \mathrm{Hom}_S(J, S/J)_0$ sending $x_{1,1} \dots x_{1,d-i} x_{2,1} \dots x_{2,i-1} x_{r,i}$ to $\mathbf{m}$ and all other generators of J to 0. Hence our φ is a linear combination of such ψ (which are evidently linearly independent). To count the number of such ψ, we see that there are $(n-2)d$ such elements $x_{1,1} \dots x_{1,d-i} x_{2,1} \dots x_{2,i-1} x_{r,i}$, and to each of them we have associated $nd - n$ monomials (hence $nd - n$ maps ψ).

In summary, there is a set of

$$2\left((nd-m)+(d-1)(nd-1)+(n-1)\frac{(d-2)(d-1)}{2}\right)+(nd-n)(n-2)d = d(d-1)(n^2+n-1) \tag{8}$$

maps in $\mathrm{Hom}_S(J, S/J)_0$ that form a k-linear spanning set. By the above constructions, they are linearly independent, so we have finished the calculation of Proposition 9.11.

We end by mentioning a question that may prove to be interesting. Besides our original question (when do polarizations of strongly stable ideals determine smooth points on their Hilbert scheme?), we may ask for properties that are preserved under further separation of a polarization, since unlike in the Artinian case, a polarization of a strongly stable ideal may be further separated (Section 3). From various computations in Macaulay2 it seems that the dimension of tangent spaces is preserved under further separations, hence the question:

Question 9.16. *Let I' be a polarization of a strongly stable ideal I, and I'' a further separation of I'. We view I' and I'' as ideals in the same polynomial ring (i.e. the ambient ring of I''). Do the tangent spaces at the points corresponding to I' and I'' (in the same Hilbert scheme H) have the same dimension?*

For instance, one sees from the definition that the pyramidal polarization $J_{n,d}$ is a separation of the standard polarization of $I_{n,d}$. Then assuming an affirmative answer to the above question, we would know that $\dim_{k(y_{n,d})} T_{y_{n,d}} = d(d-1)(n^2+n-1)$ as well, where $y_{n,d}$ is the point on the Hilbert scheme corresponding to the standard polarization.

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